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A CHARACTERIZATION OF AIGNER-CATALAN-LIKE NUMBERS

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Recently Aigner introduced a kind of infinite matrix which are called as Aigner matrix. Aigner matrix provides a common framework for a series of coefficients (called Aigner-Catalan-like numbers), with C_n and M_n as special cases. The purpose of this note is to establish a general representation of Hankel matrices of Aigner-Catalan-like numbers. As a special case, the results of Aigner are extended.

Key Words : Aigner Matrix; Aigner-Catalan-like Number; Hankel Matrix; Matrix multiplication

Definition 1 — Let $A = (a_{n,k})$ be definite matrices, indexed by $\{0, 1, 2, ...\}$, and denote by $r_m = (a_{m,0}, a_{m,1} \ldots)$ the mth row, if

(*i*) $a_{n,k} = 0$ for $n < k, a_{n,n} = g(n)$ for all n (that is, A is lower triangular),

and (*ii*) $r_m \circ r_n = a_{m+n}, {}_0$, for all m, n,

where $r_m \circ r_n = \sum_k a_{m,k} a_{n,k}$ is the inner product.

We call $A = (a_{n,k})$ as the Aigner matrix, the numbers $a_{n,0}$ of the first column will be called as Aigner-Catalan-like numbers.

Moreover, the fundamental characterization of the Aigner matrix in[1] may be restated as follows :

Proposition 1 — Let $A = (a_{n,k})$ be an the Aigner matrix with $a_{n+1\,n} = b_n$ for all n. Set $s_0 = b_0, s_1 = b_1 - b_0, \ldots, s_n = b_n - b_{n-1}, \ldots$. Then we have

$$a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + a_{n-1,k+1} \; (n \geq 1) \qquad \ldots (*)$$

$$a_{0,0} = 1, \; a_{0,k} = 0 \text{ for } k > 0.$$

Conversely, if $a_{n,k}$ is given by the recursion (*), then $(a_{n,k})$ is the Aigner matrix with $a_{n+1,n} = s_0 + \dots + s_n$.

PROOF : See Aigner[1]

According to the Proposition 1 we may (and will do so from now on) consider an Aigner matrix $(a_{n,k})$ as given by the sequence $\sigma = (s_0, s_1, \dots)$ via the recursion (*). We will then write shortly $A = A^{(\sigma)}$ and call $C_n^{(\sigma)} = A_{n,0}$ the Aigner-Catalan-like numbers of type σ. Here several special cases may be briefly mentioned (see [1]) :

σ = (1, 1, 1, ...) the Motzkin numbers M_n,

σ = (2, 2, 2, ...) the Catalan numbers C_n

and σ = (3, 3, 3, ...) the restricted hexagonal numbers H_n^6.

Let us definite t-order Hankel matrix $\tilde{C}_n^{(\sigma)}(t)$ for $C_n^{(\sigma)}$ as

$$\tilde{C}_n^{(\sigma)}(t) = \begin{pmatrix} C_t^{(\sigma)} & C_{t+1}^{(\sigma)} & \cdots & C_{n+t}^{(\sigma)} \\ C_{t+1}^{(\sigma)} & C_{t+2}^{(\sigma)} & \cdots & C_{n+t+1}^{(\sigma)} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n+t}^{(\sigma)} & B_{n+t+1}^{(\sigma)} & \cdots & B_{2n+t}^{(\sigma)} \end{pmatrix}$$

Aigner[1] obtained the computational formulas of det $C_n^{(\sigma)}(0)$ and $C_n^{(\sigma)}(1)$. The purpose of this note is to establish a general representation of Hankel matrices of Aigner-Catalan-like numbers, and extend Aigner's results.

Let $P_n, Q_n(t), J_n$ be the following $(n+1) \times (n+1)$-matrices:

$$P_n = \begin{pmatrix} a_{0,0} & 0 & 0 & \dots & 0 \\ a_{1,0} & a_{1,1} & 0 & \dots & 0 \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,0} & a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix},$$

$$Q_n(t) = \begin{pmatrix} a_{t,0} & a_{t,1} & \cdots & a_{t,t} & 0 & \cdots & 0 & 0 \\ a_{t+1,0} & a_{t+1,1} & \cdots & a_{t+1,t} & a_{t+1,t+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,t} & a_{n,t+1} & \cdots & a_{n,n-1} & a_{n,n} \\ a_{n+1,0} & a_{n+1,1} & \cdots & a_{n+1,t} & a_{n+1,t+1} & \cdots & a_{n+1,n-1} & a_{n+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{t+n,0} & a_{t+n,1} & \cdots & a_{t+n,t} & a_{t+n,t+1} & \cdots & a_{t+n,n-1} & a_{t+n,n} \end{pmatrix}$$

and $$J_n = \begin{pmatrix} s_0 & 1 & 0 & \dots & 0 & 0 \\ 1 & s_1 & 1 & \dots & 0 & 0 \\ 0, & 1 & s_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & s_{n-1} & 1 \\ 0 & 0 & 0 & \dots & 1 & s_n \end{pmatrix}$$

It is easy to obtain that

$$Q_n(0) = P_n, d_n = s_n d_{n-1} - d_{n-2} \qquad \dots (**)$$

where det $J_n = d_n$, that will be needed later.

By $$C^{(\sigma)}_{k+l} = a_{k+l,0} = r_k \circ r_l \sum_{k>0} a_{n,k} a_{l,k}$$ then we get

Theorem 1 — $\tilde{C}^{(\sigma)}_n(t) = Q_n(t) P^T_n$.

Suppose $v_{n+1}(t) = (0, \dots, 0, a_{n+1,n+1}, a_{n+2,n+1}, \dots, a_{t+n-1,n+1}, a_{t+n,n+1})$ $(t \geq 1)$ *be* $n + 1$*-dimensional vector, where* $a_{n+1,n+1}$ *occurs in the* $(n + 2 - t)$*-th place, and let*

$$R_n(t) = (0, ((v_{n+1}(t))^T)_{(n+1)\times(n+1)},$$

then we have

Theorem 2 — $Q_n(t+1) = Q_n(t) J_n + R_n(t)$.

PROOF : Apply recursion (*).

Theorem 3 — $Q_n(t) = P_n J^t_n + \sum_{k=1}^{t-1} R_n(k) J_n^{t-1-k}$

PROOF : This identity follows by repeated application of Theorem 2.

Theorem 4 — $\tilde{C}^{(\sigma)}_n(t) = \left(P_n J^t_n + \sum_{k=1}^{t-1} R_n(k) J_n^{t-1-k} \right) P^T_n$.

PROOF : Apply Theorems 1 and 3.

Corollary 1 — (Aigner[1]) $\tilde{C}^{(\sigma)}_n(0) = P_n P^T_n$,

$$\tilde{C}^{(\sigma)}_n(1) = P_n J_n P^T_n;$$

$$\det \tilde{C}^{(\sigma)}_n(0) = 1,$$

$$\det \tilde{C}^{(\sigma)}_n(1) = d_n.$$

Corollary 2 — $\tilde{C}^{(\sigma)}_n(2) = (P_n J^2_n + R_n(1)) P^T_n$.

$$\det \tilde{C}_n^{(\sigma)}(2) = \sum_{k=0}^{n} d_k^2 .$$

PROOF : The proof of first formula is straight forward by taking $t = 2$ in Theorem 4. Now we prove only the second formula. Since $\boldsymbol{P}_n \boldsymbol{J}_n^2 = \boldsymbol{Q}_n(2) - \boldsymbol{R}_n(1)$, then

$$\det J_n^2 = \det (Q_n(2) - R_n(1)).$$

Let $D_n = \det Q_n(2)$, since $\det J_n = d_n$, we get

$$D_n - D_{n-1} = d_n^2.$$

Therefore, $$D_n = \sum_{k=0}^{n} d_k^2 .$$

Hence $$\det \tilde{C}_n^{(\sigma)}(2) = \sum_{k=0}^{n} d_k^2 .$$

Example 1 — For Catalan numbers, $\sigma = (1, 2, 2, ...)$, let $\tilde{C}_n^{(\sigma)}(t) = \tilde{C}_n^{(C)}(t)$, by (**) we get $d_n = 1$, thus

$$\det \tilde{C}_n^{(C)}(0) = \det \tilde{C}_n^{(C)}(1) = 1,$$

$$\det \tilde{C}_n^{(C)}(2) = n + 1.$$

Example 2 — For Motzkin numbers, $\sigma = (1, 1, 1, ...)$, let $\tilde{C}_n^{(\sigma)}(t) = \tilde{C}_n^{(M)}(t)$, by

$$d_n = d_{n-1} - d_{n-2},$$

$$d_0 = 1, d_1 = 0.$$

Solving this recurrence relation we have

$$d_n = \frac{1}{\sqrt{3}i}\left\{\left(\frac{1-\sqrt{3}i}{2}\right)^{n-1} - \left(\frac{1+\sqrt{3}\,i}{2}\right)^{n-1}\right\} = \sum_{k=0}^{n-2} (-1)^k \binom{n-k-2}{k}.$$

Hence

$$\det \tilde{C}_n^{(M)}(0) = 1$$

$$\det \tilde{C}_n^{(M)}(1) = d_n = \frac{1}{\sqrt{3}i}\left\{\left(\frac{1-\sqrt{3}i}{2}\right)^{n-1} - \left(\frac{1+\sqrt{3}i}{2}\right)^{n-1}\right\}$$

$$= \sum_{k=0}^{n-2} (-1)^k \binom{n-k-2}{k}$$

$$\det \tilde{C}_n^{(M)}(2) = -\frac{1}{3}\sum_{k=0}^{n}\left\{\left(\frac{1-\sqrt{3}\,i}{2}\right)^{k-1} - \left(\frac{1+\sqrt{3}\,i}{2}\right)^{k-1}\right\}^2$$

$$= \frac{2}{3} + \frac{(-1)^n}{18}\left\{(3-\sqrt{3}i)\left(\frac{1+\sqrt{3}i}{2}\right)^n + (3+\sqrt{3}i)\left(\frac{1-\sqrt{3}i}{2}\right)^n\right\}.$$

Example 3 — For the restricted hexagonal numbers H_n^6, $\sigma = (3, 3, 3, \ldots)$, let $\tilde{C}_n^{(\sigma)}(t) = \tilde{C}_n^{(H)}(t)$, by (**) we have

$$d_n = 3d_{n-1} - d_{n-2},\ d_0 = 1,\ d_1 = 3.$$

Solving this recurrence relation we have

$$d_n = F_{2n+2} \text{ (the Fibonacci number)},$$

then we get $\det \tilde{C}_n^{(H)}(0) = 1,$

$$\det \tilde{C}_n^{(H)}(1) = F_{2n+2}^2$$

$$\det \tilde{C}_n^{(H)}(2) = \sum_{k=0}^{n} F_{2n+2}^2.$$

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