RN 19797/70 Regd DL-11210/92

INDIAN JOURNAL OF PURE AND APPLIED MATHEMATICS

No. 11 November 2001 Volume 32

CONTENTS

Page

A characterization of Aigner-Catalan-like numbers *by* ZHIZHENG ZHANG ... 1615

Super magic strength of a graph *by* SELVAM AVADAYAPPAN AND P. JEYANTHI ... 1621

Oscillation and nonoscillation of fourth order nonlinear neutral differential equations *by* E. THANDAPANI AND R. SAVITRI ... 1631

Hall effect on thermosolutal instability of Rivlin-Ericksen fluid with varying gravity field in porous medium *by* VEENA SHARMA AND KAMAL KISHOR ... 1643

On the limitations of the linear growth rate in Veronis' and Stern's thermohaline configurations *by* HARI MOHAN AND ANJULA ... 1659

Local automorphisms of nest algebras *by* CHENGJUN HOU AND SHENGZHAO HOU ... 1667

Some fixed point theorems of increasing operators and applications *by* QIU QIUSHENG ... 1679

On generalized statistically convergent sequences *by* BINOD CHANDRA TRIPATHY AND MAUSUMI SEN ... 1689

Periodic boundary value problems and monotone iterative methods for first order impulsive differential equations with delay *by* FENGQIN ZHANG, ZHIEN MA AND JURANG YAN ... 1695

Variational Lyapunov method and stability theory *by* XILIN FU AND WEIJIE FENG ... 1709

Connectedness in Fuzzy suprabitopological spaces *by* S. SAMPATH KUMAR ... 1725

Controller design for a nonlinear servomechanism in the presence of system uncertainties using multilayered neural network *by* C. MOHAN, SHIV PRASAD YADAV, SURENDRA KUMAR AND RADHI A ZABOON ... 1729

Functional central limit theorems for iterated function systems controlled *by* regenerative sequences *by* O. LEE AND D. W. SHIN ... 1749

Indian J. pure appl. Math., 32(11) : 1621-1630, November 2001

SUPER MAGIC STRENGTH OF A GRAPH

SELVAM AVADAYAPPAN*,

Department of Mathematics, Dr Sivanthi Aditanar College of Engineering, Tiruchendur 628 215, *India*

P. JEYANTHI

Department of Mathematics, Govindammal Aditanar College for Women, Tiruchendur 628 215, *India*

AND

R. VASUKI

Department of Mathematics, Dr. Sivanthi Aditanar College of Engineering, Tiruchendur 628 215, *India*

(*Received* 15 *November* 1999; *accepted* 8 *January* 2001)

Let G be a graph with v vertices and ε edges. G is said to be magic if there exists a bijection $f: V \cup E \to \{1, 2, 3, ..., v + \varepsilon\}$ such that for all edges xy, $f(x)+f(y)+f(xy)$ is a constant which is denoted by $c(f)$. Such a bijection is called a magic labeling of G. A magic labeling f of G is called a super magic labeling if $f(V) = \{1, 2, 3, ..., v\}$. In this paper, the concept of super magic strength of a graph is introduced. The super magic strength of a graph G is defined as the minimum of all $c(f)$ where the minimum runs over all super magic labelings of G and is denoted by $sm(G)$. The exact value of super magic strength of some well-known graphs are obtained in this paper.

Key Words : Graph Labeling; Magic Labeling; Magic Strength; Super Magic Labeling; Super Magic Strength

1. INTRODUCTION

In this paper, we consider only finite simple undirected graphs. Our notations and terminology are as in[2]. In particular, $\varepsilon(G)$ (or simply ε) denotes the number of edges in G. The graph P_n is the path on n vertices where as C_n is the cycle on n vertices. $B_{n,n}$ is the graph obtained from two copies of $K_{1,n}$ by joining the vertices of maximum degree by an edge which is called an n-bistar.

In 1970, Kotzig and Rosa[7] defined a *magic labeling* of a graph $G(V, E)$ is a bijection $f: V \cup E \to \{1, 2, 3, ..., v+\varepsilon\}$ such that for all edges $xy, f(x)+f(y)+f(xy)$ are the same.

*Present address: Department of Mathematics, VHNSN College, Virudhunagar 626 001, India.

For example, the magic labeling of some graphs are shown in Fig. 1.

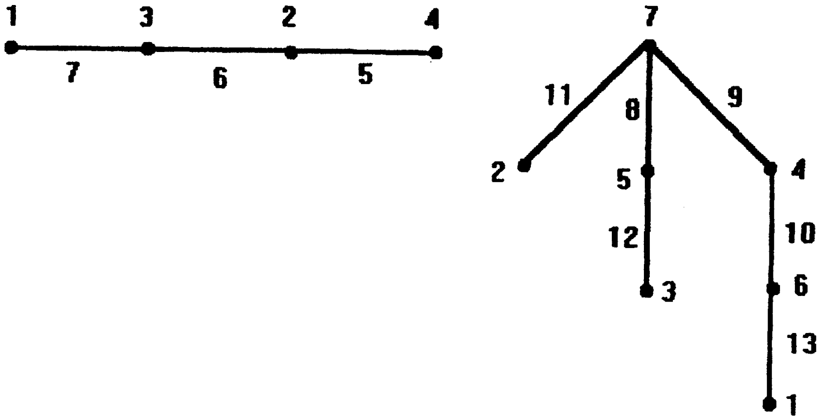

FIG. 1

A graph G is said to be *magic* if it has a magic labeling. Ringel and Llado[8] called this graph *edge-magic*. They have proved that if G is a graph in which both v and ε are even such that $v + \varepsilon \equiv 2 \pmod 4$ and in which each vertex has odd degree, then G is not magic. Also it was shown in[8] that each caterpillar is magic.

In[7], the following results have been proved :

1. $K_{m,n}$ is magic for all m and n.

2. C_n is magic for all $n \geq 3$.

3. nP_2 (the disjoint union of n copies of P_2) is magic if and only if n is odd.

The following conjecture has been raised in[7]:

Conjecture 1 — Every tree is magic.

Many more results, conjectures and open problems on graph labeling have been discussed in[4&5]

Recently, S. Avadayappan *et al.*[1] introduced the concept of magic strength of a graph. We know that for any magic labeling f of G, there is a constant $c(f)$ such that $f(x) + f(y) + f(xy) = c(f)$ for any edge $xy \in E(G)$. *The magic strength of* G, $m(G)$ is defined as the minimum of all $c(f)$ where the minimum is taken over all magic labelings of G. That is, $m(G) = \min \{c(f) : f$ is a magic labeling of $G.\}$

In[1], the following results have been obtained :

1. $m(P_{2n}) = 5n + 1$, $m(P_{2n+1}) = 5n + 3$.

2. $m(K_{1,n}) = 2n + 4$.

3. $m(B_{n,n}) = 5n + 6$.

4. $m((2n + 1) P_2) = 9n + 6$.

5. $m(\langle K_{1,n} : 2 \rangle) = 4n + 9$.

6. If a tree T is k-sequential then T is magic.

A magic labeling of a graph $G(V, E)$ is called a *super magic labeling* of G if $f(V) = \{1, 2, 3, ..., v\}$ and thus $f(e) = \{v+1, v+2, ..., v+\varepsilon\}$.

For example, the super magic labelings of some graphs are shown in Fig. 2.

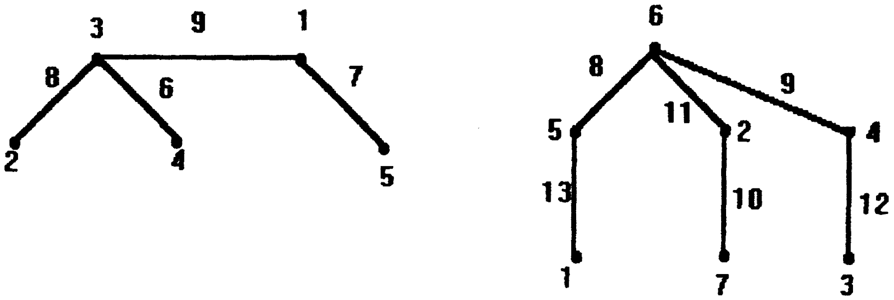

FIG. 2

A graph is said to be *super magic* if it has a super magic labeling.

In[3], the following results have been proved.

1. A cycle C_n is super magic if and only if n is odd.

2. A complete bipartite graph $K_{m,n}$ is super magic if and only if $m = 1$ or $n = 1$.

The following conjecture has been raised in[3] :

Conjecture 2 — Every tree is super magic.

In this paper, we introduce the concept of super magic strength of a graph. The super magic strength of a graph G, $sm(G)$ is defined as the minimum of all $c(f)$ where the minimum is taken over all super magic labelings f of G. That is,

$sm(G) = \min\{c(f) : f \text{ is a super magic labeling of } G\}$.

One can easily note that, since the labels are from the set $\{1, 2, 3, ..., v+\varepsilon\}$,

$$v+\varepsilon+3 \leq sm(G) \leq 3v.$$

In this paper, we establish the super magic strength of some well-known graphs.

To proceed further, we make the following note :

Note 1 — Let f be a super magic labeling of G with the constant $c(f)$. Then adding all the constants obtained at each edge, we get

$$\varepsilon\, c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e).$$

2. Super Magic Strength of Some Trees

In this section, we obtain the super magic strength of the path P_n on n vertices, the n-bistar obtained from two disjoint copies of $K_{1,n}$ by joining the center vertices by an edge and the tree $\langle K_{1,n} : 2 \rangle$ obtained from $B_{n,n}$ by subdividing the middle edge with a new vertex.

Lemma 1 — $sm(P_{2n}) \leq 5n+1$ and $sm(P_{2n+1}) \leq 5n+3$.

PROOF : We prove this lemma by assigning super magic labeling to P_{2n} and P_{2n+1}. Let $v_1, v_2, v_3, \ldots, v_{2n}$ be the consecutive vertices and $e_1, e_2, e_3, \ldots, e_{2n-1}$ be the consecutive edges of P_{2n} such that $e_i = v_i v_{i+1}$, for $1 \le i \le 2n-1$. Then the following labeling f is a super magic labeling of P_{2n} :

$$f(v_{2i}) = n+i, f(v_{2i-1}) = i \text{ for } 1 \le i \le n \text{ and } f(e_i) = 4n - i \text{ for } 1 \le i \le 2n-1.$$

Similarly, we define a super magic labeling g of P_{2n+1} as follows :

$$g(v_{2i}) = i \text{ for } 1 \le i \le n, g(v_{2i-1}) = n+i \text{ for } 1 \le i \le n+1 \text{ and } g(e_i) = 4n+2-i$$

for $1 \le i \le 2n$.

Thus $sm(P_{2n}) \le 5n+1$ and $sm(P_{2n+1}) \le 5n+3$.

For example, the super magic labelings of P_8 and P_9 are shown in Fig. 3. ■

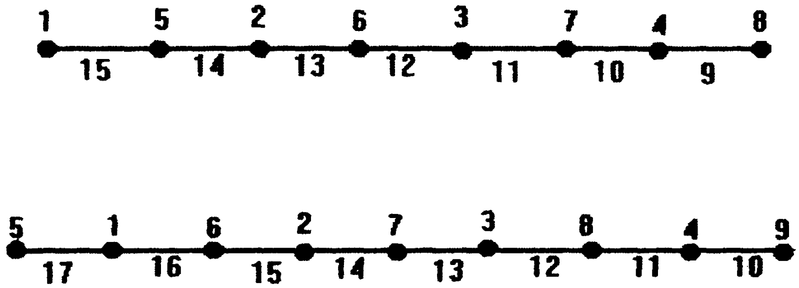

FIG. 3

Lemma 2 — $sm(P_{2n}) \ge 5n+1$ and $sm(P_{2n+1}) \ge 5n+3$.

PROOF : Here $\varepsilon = 2n-1$ and $v + \varepsilon = 4n-1$. Also by Note 1, if f is a super magic labeling of P_{2n} with constant $c(f)$, then

$$\varepsilon\, c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e).$$

That is, $$(2n-1)\, c(f) = \sum_{i=2}^{2n-1} 2f(v_i) + f(v_1) + f(v_{2n}) + \sum_{i=1}^{2n-1} f(e_i)$$

$$= \sum_{i=1}^{2n} f(v_i) + \sum_{i=1}^{2n-1} f(e_i) + \sum_{i=2}^{2n-1} f(v_i)$$

$$= 1 + 2 + \ldots + 4n - 1 + \sum_{i=2}^{2n-1} f(v_i)$$

$$= 2n\,(4n-1) + \sum_{i=2}^{2n-1} f(v_i).$$

Therefore, $$c\,(f) = (4n-1)\,(2n)/(2n-1) + \sum_{i=2}^{2n-1} f(v_i)/(2n-1)$$

$$> 4n - 1 + 2 + (1 + 2 + 3 + \ldots + 2n - 2)/(2n-1)$$

$$= (4n-1) + 2 + (n-1)$$

$$= 5n.$$

Thus $c(f) > 5n$ and hence, $c(f) \geq 5n + 1$ which implies that $sm\,(P_{2n}) \geq 5n + 1$.

Similarly, we can prove that $Sm\,(P_{2n+1}) \geq 5n + 3$. ■

Combining Lemmas 1 and 2, we can state that

Theorem 1 — *$sm\,(P_{2n}) = 5n + 1$ and $sm\,(P_{2n+1}) = 5n + 3$.* ■

Theorem 2 — *$sm\,(B_{n,\,n}) = 5n + 6$.*

PROOF : Let $V(B_{n,\,n}) = \{u, v; u_1, u_2, u_3, \ldots, u_n; v_1, v_2, v_3, \ldots, v_n\}$ and $E(B_{n,\,n}) = \{uv, uu_i, vv_i : : 1 \leq i \leq n\}$.

First, we show that $sm\,(B_{n,\,n}) \leq 5n + 6$ by giving a super magic labeling of $B_{n,\,n}$. Consider the following labeling f of $B_{n,\,n}$:

$f(u) = 1$, $f(v) = 2n + 2$, $f(u_i) = n + 1 + i$, $f(v_i) = i + 1$ for $1 \leq i \leq n$, $f(uv) = 3n + 3$, $f(uu_i) = 4n + 4 - i$ and $f(vv_i) = 3n + 3 - i$ for $1 \leq i \leq n$.

One can easily verify that f is a super magic labeling of $B_{n,\,n}$ with $c(f) = 5n + 6$ and hence $sm\,(B_{n,\,n}) \leq 5n + 6$.

For example, a super magic labeling of $B_{5,\,5}$ is shown in Fig. 4.

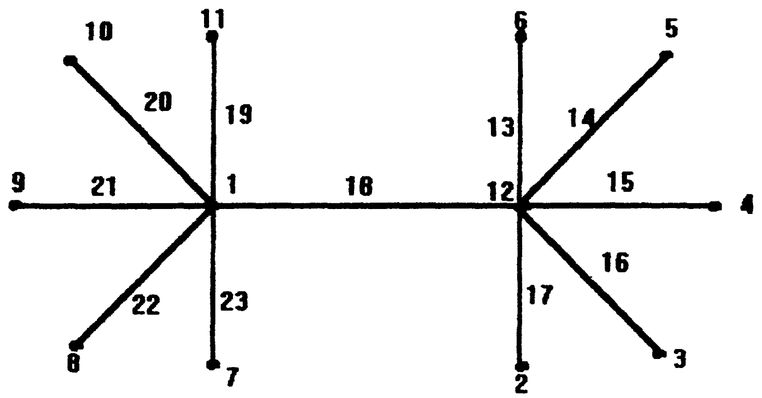

FIG. 4

Now it is enough to prove that $sm\,(B_{n,\,n}) \geq 5n + 6$. Let f be a super magic labeling of $B_{n,\,n}$ with constant $c(f)$. Then by Note 1,

$$\varepsilon\, c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e).$$

That is, $$(2n+1)\, c(f) = \sum_{i=1}^{n} f(u_i) + \sum_{i=1}^{n} f(v_i) + (n+1) f(u) + (n+1) f(v)$$

$$+ \sum_{e \in E} f(e) = \sum_{v \in V} f(v) + \sum_{e \in E} f(e) + nf(u) + nf(v)$$

$$= n(f(u) + f(v)) + 1 + 2 + 3 + \ldots + 4n + 3.$$

Therefore, $c(f) = (4n + 3)(2n + 2)/(2n + 1) + (n(f(u) + f(v)))/(2n + 1)$

$$= 4n + 5 + (1 + n(f(u) + f(v)))/(2n + 1).$$

Since $c(f)$ is an integer, $(nf(u) + nf(v) + 1)/(2n + 1)$ must also be an integer.

That is, $nf(u) + nf(v) + 1 \equiv 0 \bmod (2n + 1)$ and hence $nf(u) + nf(v) \equiv 2n \bmod (2n + 1)$ which implies that $f(u) + f(v) \equiv 2 \bmod (2n + 1)$. But $f(u) + f(v) \geq 3$ and thus $f(u) + f(v) \geq 2n + 3$.

Therefore, $c(f) \geq 4n + 5 + (n(2n + 3) + 1)/(2n + 1) = 5n + 6$. This shows that

$sm\,(B_{n,n}) \geq 5n + 6$. Hence, $sm\,(B_{n,n}) = 5n + 6$. ■

Recall that the tree $\langle K_{1,n} : 2 \rangle$ is obtained from the n-bistar $B_{n,n}$ by subdividing the middle edge uv with a new vertex w. The super magic strength of $\langle K_{1,n} : 2 \rangle$ is established in the following theorem.

Theorem 3 — $sm\,\langle K_{1,n} : 2 \rangle) = 4n + 9.$

PROOF : First, we prove that $\langle K_{1,n} : 2 \rangle$ is super magic. Define a labeling f on $\langle K_{1,n} : 2 \rangle$ as follows :

$f(u) = 1$, $f(v) = 2$, $f(w) = n + 3$, $f(u_i) = i + 2$ and $f(v_i) = n + 3 + i$ for $1 \leq i \leq n$, $f(uw) = 3n + 5$, $f(vw) = 3n + 4$, $f(uu_i) = 4n + 6 - i$ and $f(vv_i) = 3n + 4 - i$ for $1 \leq i \leq n$.

One can check that the above labeling f is a super magic labeling of $\langle K_{1,n} : 2 \rangle$ with $c(f) = 4n + 9$. Thus, $sm\,(\langle K_{1,n} : 2 \rangle) \leq 4n + 9$.

For example, a super magic labeling of $\langle K_{1,3} : 2 \rangle$ is shown in Fig. 5.

Now we show that $sm\,(\langle K_{1,n} : 2 \rangle) \geq 4n + 9$. Let f be a super magic labeling of $\langle K_{1,n} : 2 \rangle$ with constant $c(f)$. Then by Note 1,

$$\varepsilon\, c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e).$$

Here, $\varepsilon = 2n + 2$. Therefore,

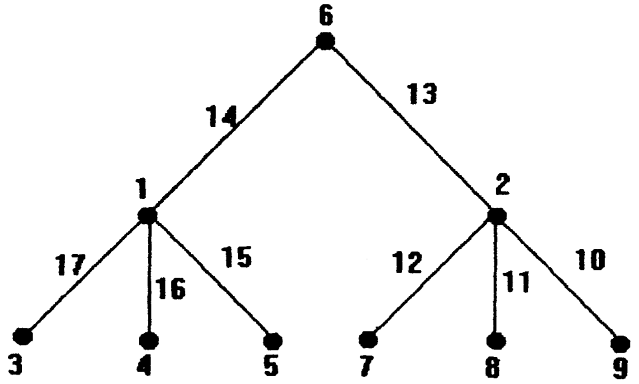

FIG. 5

$$(2n+2)\, c(f) = \sum_{i=1}^{n} f(u_i) + \sum_{i=1}^{n} f(v_i) + (n+1) f(u) + (n+1) f(v) + 2 f(w) + \sum_{e \in E} f(e)$$

$$= \sum_{v \in V} f(v) + \sum_{e \in E} f(e) + n(f(u) + f(v)) + f(w)$$

$$= (1 + 2 + 3 + \ldots + 4n + 5) + n\,(f(u) + f(v)) + f(w)$$

$$= (4n+5)\,(4n+6)/2 + n\,(f(u) + f(v)) + f(w).$$

This implies that

$$c\,(f) = (4n+5)\,(4n+6)/2\,(2n+2) + (n\,(f(u) + f(v)) + f(w))/(2n+2)$$

$$= 4n + 7 + (n\,(f(u) + f(v)) + f(w) + 1)/(2n+2)$$

$\geq 4n + 9$ (since $f(u) + f(v) \geq 3$ and since $c\,(f)$ is an integer).

Thus $sm\,(\langle K_{1,n} : 2 \rangle) \geq 4n + 9$ which gives that $sm\,(\langle K_{1,n} : 2 \rangle) = 4n + 9$. ■

3. SUPER MAGIC STRENGTH OF C_{2n+1}

It has been proved in[3] that C_n is super magic if and only if n is odd. In this section, we establish the super magic strength of C_{2n+1}.

Theorem 5 — $sm\,(C_{2n+1}) = 5n + 4$.

PROOF : Let C_{2n+1} be $v_1 v_2 \ldots v_{2n+1} v_1$. Then the following labeling f is a super magic labeling of C_{2n+1} :

$f(v_{2i+1}) = i + 1$ for $0 \leq i \leq n, f(v_{2i}) = n + i + 1$ for $1 \leq i \leq n, f(v_i v_{i+1}) = 4n + 2 - i$

for $1 \leq i \leq 2n$ and $f(v_{2n+1} v_1) = 4n + 2$.

For example, a super magic labeling of C_7 is shown in Fig. 6.

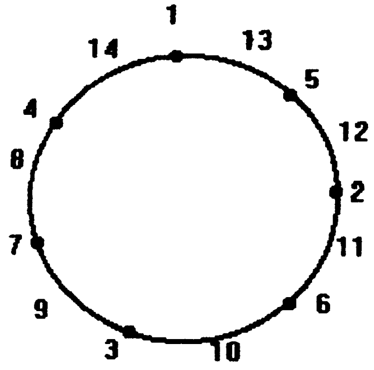

FIG. 6

Thus $$sm\,(C_{2n+1}) \leq 5n + 4.$$

Now by Note 1, if f is a super magic labeling of C_{2n+1}, then

$$(2n+1)\, c\,(f) = \sum_{v \in V} 2f(v) + \sum_{e \in E} f(e)$$

$$= \sum_{v \in V} f(v) + \sum_{e \in E} f(e) + \sum_{v \in V} f(v)$$

$$= 1 + 2 + \ldots + (4n+2) + \sum_{v \in V} f(v)$$

$$\geq (4n+2)\,(4n+3)/2 + (1 + 2 + \ldots + (2n+1))$$

$$= (2n+1)\,(4n+3) + (2n+1)\,(n+1)$$

$$= (5n + 4)\,(2n + 1).$$

Therefore, $c(f) \geq 5n + 4$. This implies that $sm\,(C_{2n+1}) \geq 5n + 4$ and hence

$$sm(C_{2n+1}) = 5n + 4.$$ ■

4. SOME OBSERVATIONS

We conclude this paper with some observations.

Observation 1 — $sm\,(K_{1,n}) = 2n + 4$.

For example, a super magic labeling of $K_{1,5}$ is shown in Fig. 7.

Observation 2 — $sm\,(P_n^2) = 3n$.

For, let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of P_n^2. Then $E\,(P_n^2) = \{v_i v_{i+1} : 1 \leq i \leq n-1;\ v_i v_{i+2} : 1 \leq i \leq n-2\}$. A super magic labeling of P_n^2 is given below :

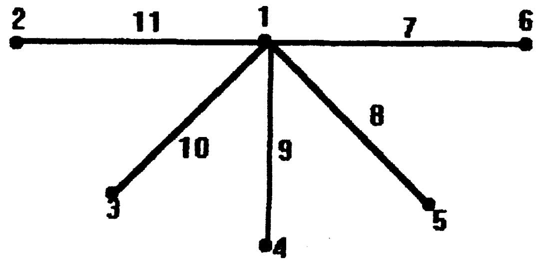

FIG. 7

$f(v_i) = i$ for $1 \le i \le n, f(v_i v_{i+1}) = 3n - (2i+1)$ for $1 \le i \le n-1$, and $f(v_i v_{i+2}) = 3n - (2i+2)$ for $1 \le i \le n-2$.

Therefore, $sm(P_n^2) \le 3n$. But since $\varepsilon(P_n^2) = 3n - 3$, we have $sm(P_n^2) \ge 3n$. Thus $sm(P_n^2) = 3n$.

For example, a super magic labeling of P_5^2 is shown in Fig. 8.

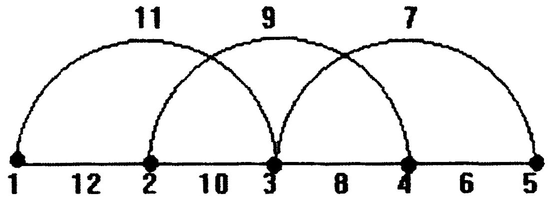

FIG. 8

Observation 3 — $sm((2n+1)P_2) = 9n+6$. (Note that nP_2 is super magic if and only if n is odd.)

For, let the vertices of $(2n+1)P_2$ be $u_1, u_2, u_3, \ldots, u_{2n+1}; v_1, v_2, v_3, \ldots, v_{2n+1}$ and let the edge set be $\{u_i v_i : 1 \le i \le 2n+1\}$. Define $f: V \cup E \to \{1, 2, 3, \ldots, 6n+3\}$ in such a way that $f(u_i) = i$ for $1 \le i \le 2n+1$, $f(v_i) = 4n+3-2i$ for $1 \le i \le n, f(v_{n+i}) = 4n+4-2i$ for $1 \le i \le n+1$, $f(u_i v_i) = 5n+3+i$ for $1 \le i \le n$ and $f(u_{n+i} v_{n+i}) = 4n+2+i$ for $1 \le i \le n+1$.

It is easy to check that f is a super magic labeling of $(2n+1)P_2$ with $c(f) = 9n+6$. Thus $(2n+1)P_2$ is super magic.

For example, a super magic labeling of $7P_2$ is shown in Fig. 9.

It remains to show that $sm((2n+1)P_2) = 9n+6$. Let f be a super magic labeling of $(2n+1)P_2$ with constant $c(f)$. Then by Note 1, we have

$$\varepsilon\, c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e).$$

That is,
$$(2n+1)\, c(f) = \sum_{v \in V} f(v) + \sum_{e \in E} f(e)$$

$$= 1 + 2 + 3 + \ldots + (6n+3)$$

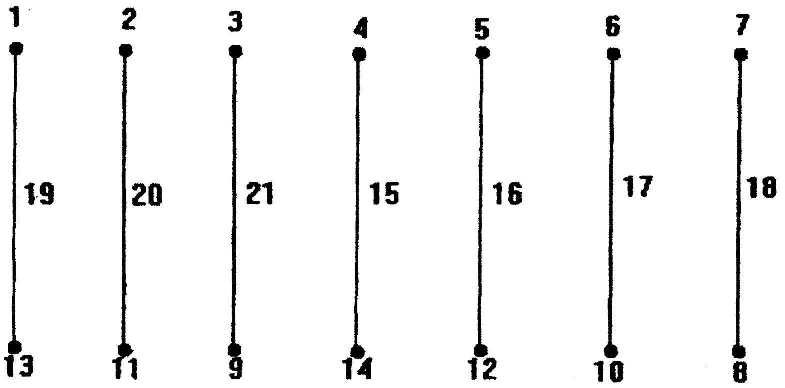

FIG. 9

$$= (6n+3)(6n+4)/2$$

and hence $c(f) = 9n + 6$.

This is true for any magic labeling f of $(2n + 1)\ P_2$. Therefore, $sm\ ((2n+1)\ P_2) = 9n+6$.

REFERENCES

1. Selvam Avadayappan, P. Jeyanthi and R. Vasuki, *Indian J. pure appl. Math.* **31** (2000) 873-85.
2. J. A. Bondy and U.S.R. Murty, *Graph Theory With Applications*, Macmillan, London, 1976.
3. H. Enomotoana, A. S. Llado, *SVT J. Math.* (1998) 105-109.
4. J. A. Gallian, *J. Graph Theory* **13** (1989) 491-504.
5. J. A. Gallian, *Discrete appl. Math.* **49** (1994) 213-29.
6. J. A. Gallian, *Electron J. Graph Theory* **7** (1985) 195-201.
7. A. Kotzig and A. Rosa, *Canad. math. Bull.* **13** (1970) 451-61.
8. G. Ringel and A. S. Llado, *Bull. ICA* **18** (1996) 83-85.