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OSCILLATION AND NONOSCILLATION OF FOURTH ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Some oscillation and nonoscillation criteria for the fourth order neutral differential equation

$$(r(t)\,(y(t)+p(t)\,y(h(t))''\,)''+f\,(t,\,y\,(\sigma\,(t)))=0$$

are established. These results generalize some known results for ordinary and delay differential equations. Examples are inserted to illustrate the results.

Key Words : Neutral Differential Equation; Nonoscillation; Oscillation

INTRODUCTION

The oscillatory and asymptotic behaviour of solutions of fourth order differential equations of the form

$$(r(t)\,y''\,(t))''+f\,(t,\,y(\sigma\,(t)))=0,\ t\ge t_0 \qquad \text{... }(A)$$

has been discussed to some extent in a number of studies, see for example[1, 2, 5-8, 11-32] and the references cited therein. Beginning with Atkinson's fundamental paper[3] much of the work has been devoted to establishing necessary and sufficient conditions for all solutions of the equation under consideration to be oscillatory, and for the converse the existence of nonoscillatory solutions. Following this trend, in this paper we present some new oscillation and nonoscillation results for the fourth order nonlinear neutral differential equation

$$(r(t)\,(y(t)+p(t)y(h(t))'')''+f(t,\,y(\sigma\,(t))=0,\ t\ge t_0\ge 0 \qquad \text{... }(E)$$

where the following conditions are always assumed to hold :

(C_1) $r(t)$ is positive and continuous for $t \ge t_0$, and

$$\int_{t_0}^{\infty}\frac{t}{r(t)}\,dt=\infty;$$

(C_2) $p(t)$ is continuous for $t \geq t_0$ and $0 \leq p(t) \leq p < 1$;

(C_3) $h(t) \leq t$ and $\sigma(t)$ are continuous for $t \geq t_0$ and

$$\lim_{t \to \infty} h(t) = \lim_{t \to \infty} \sigma(t) = \infty; \text{ and}$$

(C_4) $f: [t_0, \infty) \times I\!R \to I\!R$ is continuous and $f(t, y)$ is nondecreasing in y with $yf(t, y) > 0$, $y \neq 0$ for each fixed $t \geq t_0$.

We restrict our discussion to those solutions $y(t)$ of eq. (E) which exists on some half line $[T_y, \infty)$, $T_y \in I\!R$ and satisfy $\sup \{|y(t)|; t \geq T\} > 0$ for every $T \in [T_y, \infty)$. Such a solution is said to be oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory.

In Section 1, we present some lemmas, which are, needed, for our subsequent discussion. In Section 2 we establish necessary and sufficient conditions for eq. (E) to have nonoscillatory solutions with specified asymptotic behaviour and finally in Section 3, we provide conditions for the oscillation of all solutions of eq. (E). The results obtained here include previously known result result of Terry and Wong[31] and Kusano and Naito[14 & 16] for ordinary and delay differential equations of the form (A). For more recent results about neutral differential equations see the monographs of Bainov and Mishev[4] and Györi and Ladas[9] .

1. Some Preliminary Lemmas

In this section we state and prove some lemmas, which are useful in establishing our main results. Here and throughout we use the notation :

$$Z(t) = y(t) + p(t)y(h(t))$$

$$R(t) = \int_{t_0}^{t} \int_{t_0}^{s} \left(\frac{\tau}{r(\tau)} d\tau \, ds \right)$$

and $$R(t, T) = \int_{T}^{t} \int_{T}^{S} \frac{(\tau - T)}{r(\tau)} d\tau \, ds.$$

Note that if $y(t)$ is eventually positive or eventually negative then by condition (C_2) the function $Z(t)$ is also eventually positive or eventually negative, respectively. We state and prove lemmas for the case $y(t)$ is eventually positive since the case $y(t)$ eventually negative is similar.

Lemma 1.1 — If $y(t)$ is an eventually positive solution of eq. (E), then there are only the following two cases for t large enough :

(I) $Z(t) > 0, Z'(t) > 0, r(t) Z''(t) > 0, (r(t) Z''(t))' > 0$; and

(II) $Z(t) > 0, Z'(t) > 0, r(t) Z''(t) < 0, (r(t) Z''(t))' > 0$.

Lemma 1.2 — If $T \geq t_0$, then $\lim_{t \to \infty} \frac{R(t, T)}{R(t)} = 1$.

Lemma 1.3 — If $y(t)$ is an eventually positive solution of eq. (E) then there exist $T \geq t_0$ and $k > 0$ such that

$$\frac{1}{2}(r(t)\,Z''(t))'\,R(t) \leq Z(t) \leq kR(t) \text{ for } t \geq T. \quad \ldots (1)$$

Lemma 1.4 — Let $y(t)$ be an eventually positive solution of eq. (E), then there exist $t_1 \geq t_0$ such that for any $T \geq t_1$ we have

$$Z(t) \geq \int_T^t R(s, T) f(s, y(\sigma(s)))\, ds \text{ for } t \geq T. \quad \ldots (2)$$

The proofs of Lemma 1.1 to 1.4 can be modelled as that of in[14&18] and hence the details are omitted.

Lemma 1.5 — If $y(t)$ is an eventually positive solution of eq. (E), then there exists $T \geq t_0$ such that

$$(1 - p(t))\, Z(t) \leq y(t) \leq Z(t) \text{ for } t \geq T. \quad \ldots (3)$$

PROOF : Let $y(t)$ be an eventually positive solution of eq. (E) for $t \geq T$. Then from the definition of $Z(t)$, we have $Z(t) \geq y(t)$ for $t \geq T$. From Lemma 1.1 we have $Z(t) > 0$ and $Z'(t) > 0$ for $t \geq T$. Hence,

$$Z(t) - p(t)\, Z(h(t)) = y(t) - p(t)\, p(h(t)) y(h(h(t))) \leq y(t)$$

or

$$(1 - p(t))\, Z(t) \leq Z(t) - p(t)\, Z(h(t)) \leq y(t) \text{ for } t \geq T.$$

This complete the proof of the lemma.

Lemma 1.6 — If $y(t)$ is an eventually positive solution of eq. (E) then there exists $T \geq t_0$ such that

$$Z'(t) \geq \frac{1}{2}(r(t)\, Z''(t))'\, R'(t) \text{ for } t \geq T.$$

Also if

$$\sigma(t) \leq t, \text{ then } Z'(\sigma(t)) \geq \frac{1}{2}(r(t)\, Z''(t))'\, R'(\sigma(t)) \text{ for } t \geq T. \quad \ldots (4)$$

PROOF : From Lemma 1.1 we have for $t \geq t_1 \geq t_0$, $Z(t) > 0, Z'(t) > 0, (r(t)\, Z''(t))' > 0$ and $(r(t)\, Z''(t))'' < 0$. Hence,

$$Z'(t) \geq \int_{t_1}^t Z''(s)\, ds = \int_{t_1}^t \frac{1}{r(s)} r(s)\, Z''(s) ds$$

$$\geq \int_{t_1}^t \frac{1}{r(s)} \left(\int_{t_1}^s r(\tau)\, Z''(\tau))'\, d\tau \right) ds$$

$$\geq (r(t)\,Z''(t))' \int_{t_1}^{t} \frac{s-t_1}{r(s)}\,ds = (r(t)\,Z''(t))'\,R'(t, t_1).$$

From Lemma 1.2 we conclude that there exists $T \geq t_1$ such that

$$R'(t, t_1) \geq \frac{1}{2} R'(t) \text{ for } t \geq T$$

and hence $$Z'(t) \geq \frac{1}{2} (r(t)\,Z''(t))'\,R'(t) \text{ for } t \geq T.$$

Since $(r(t)\,Z''(t))'' < 0$ and $\sigma(t) \leq t$, we have

$$Z'(\sigma(t)) \geq \frac{1}{2} (r(\sigma(t))\,Z''(\sigma(t)))'\,R'(\sigma(t)) \geq \frac{1}{2}(r(t)\,Z''(t))'\,R'(\sigma(t)),\ t \geq T.$$

The proof is now complete.

2. Existence of Nonoscillatory Solutions

In this section we give criteria for the existence of nonoscillatory solutions of eq. (E) with some specific asymptotic behaviour.

Theorem 2.1 — *A necessary and sufficient condition for the eq.* (*E*) *to have a non oscillatory solution* $y(t)$ *such that* $\lim_{t \to \infty} \frac{Z(t)}{R(t)} = \alpha \neq 0$ *is that*

$$\int_{t_0}^{\infty} f(t, c\,(1 - p(\sigma(t)))\,R(\sigma(t)))\,dt < \infty \qquad \text{... (5)}$$

for some $c \neq 0$.

PROOF : *Necessity* : Let $y(t)$ be a nonoscillatory solution of (E) such that $\lim_{t \to \infty} \frac{Z(t)}{R(t)} = \alpha \neq 0$.

We may assume that $y(t)$ is eventually positive. Then there exist constants $T \geq t_0$, $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$\alpha_1 R(\sigma(t)) \leq Z(\sigma(t)) \leq \alpha_2 R(\sigma(t)) \text{ for } t \geq T. \qquad \text{... (6)}$$

In view of (3) and (6), we obtain

$$y(\sigma(t)) \geq (1 - p(\sigma(t)))\,\alpha_1 R(\sigma(t)) \text{ for } t \geq T. \qquad \text{... (7)}$$

Since $(r(t)\,Z''(t))' > 0$ by Lemma 1.1, on integrating (E), we have

$$\int_{T}^{\infty} f(t, y(\sigma(t))\,dt < \infty. \qquad \text{... (8)}$$

From (7) and (8) we conclude that

$$\int_T^{\infty} f(t, \alpha_1 (1 - p(\sigma(t)) R(\sigma(t))) \, dt < \infty.$$

Sufficiency — Suppose that (5) holds for some $c \neq 0$. We may assume that $c > 0$ since a similar argument holds if $c < 0$. Let $d > 0$ be such that $\frac{4d}{1-p} < c$ and choose $T \geq t_0$ so large that

$$\int_T^{\infty} f(t, c(1 - p(\sigma(t))) R(\sigma(t))) \, dt < \frac{(1-p)\, d}{8}$$

and
$$T_0 = \min \left\{ T, \inf_{t \geq T} h(t), \inf_{t \geq T} \sigma(t) \right\} \geq t_0.$$

Let $C[T_0, \infty)$ be the locally convex space of all continuous functions on $[T_0, \infty)$ with the topology of uniform convergence on any compact subintervals of $[T_0, \infty)$. Define a closed, convex subset Y of $C[T_0, \infty)$ by

$$Y = \{ y \in C[T_0, \infty] : 2(1-p)\, d R(t) \leq y(t) \leq 4d R(t) \text{ on } [T, \infty) \text{ and } y(t) = 0 \text{ on } [T_0, T] \},$$

Now we define an operator $F : Y \to C[T_0, \infty)$ by

$$(Fy)(t) = \begin{cases} (3+p) d R(t) - p(t) y(h(t)) + \displaystyle\int_T^t \int_T^{s_1} \frac{1}{r(s_2)} \int_T^{s_2} \int_{s_1}^{\infty} f(s, y(\sigma(s))) \, ds \, ds_1 \, ds_2 \, ds_3, & t \geq T \\ 0, \quad T_0 \leq t \leq T. \end{cases}$$

It is a matter of routine calculation to verify that F is continuous mapping which sends Y into relatively compact subset of Y. Therefore, the Schauder-Tychonoff fixed point theorem ensures the existence of an element $y \in Y$ such that $Fy = y$. It is easy to see that $y(t)$ is a solution of (E) for $t \geq T_0$. Since, by L'Hospitals' rule

$$\lim_{t \to \infty} \frac{Z(t)}{R(t)} = \lim_{t \to \infty} \frac{Z'(t)}{R'(t)} = \lim_{t \to \infty} \frac{r(t) Z''(t)}{m} r(t) R''(t) = \lim_{t \to \infty} \frac{r(t) Z''(t)}{t}$$

$$= \lim_{t \to \infty} (r(t) Z''(t))' = (3+p)\, d,$$

it turns out that $y(t)$ is a nonoscillatory solution of (E) with the desired asymptotic property. The proof is now complete.

***Theorem* 2.2** — *A necessary and sufficient condition for eq. (E) to have a nonoscillatory solution y(t) such that* $\lim_{t \to \infty} Z(t) = \beta \neq 0$ *is that*

$$\int_{t_0}^{\infty} R(t)\,|f(t, c(1-p(\sigma(t)))|\,dt < \infty \qquad \text{... (9)}$$

for some $c \neq 0$.

PROOF : *Necessity* : Let $y(t)$ be a nonoscillatory solution of (E) such that $\lim_{t\to\infty} Z(t) = \beta \neq 0$.

We may assume that $y(t)$ is eventually positive. Then there exist constants $T \geq t_0$, $\beta_1 > 0$, $\beta_2 > 0$ for which $\beta_1 \leq Z(\sigma(t)) \leq \beta_2$ for $t \geq T$. Hence, from (3) we have

$$y(\sigma(t)) \geq \beta_1 (1 - p(\sigma(t))) \text{ for } t \geq T. \qquad \text{... (10)}$$

Multiply eq. (E) by $R(t)$ and integrate it from T to t, we obtain

$$\int_T^t R(s) f(s, y(\sigma(s)) ds = -\int_T^t R(s)\,(r(s)\,Z''(s))''\,ds$$

$$= -R(t)\,(r(t)\,Z''(t))' + R'(t)\,r(t)\,Z''(t) - t\,Z''(t) + Z(t) + k, \qquad \text{... (11)}$$

where k is a constant. Observe that $Z(t)$ is subject to the case (II) of Lemma 1.1, we deduce from (11) that

$$\int_T^{\infty} R(t) f(t, y(\sigma(t)))\,dt < \infty. \qquad \text{... (12)}$$

From (10) and (12) we obtain

$$\int_T^{\infty} R(t) f(t, \beta_1 (1 - p(\sigma(t)))\,dt < \infty.$$

Sufficiency — Suppose that (9) holds for some constant $c > 0$. The case of negative c can be treated similarly. Let $d \leq \frac{c(1-p)}{2}$ and take $T \geq t_0$ so large that

$$\int_T^{\infty} R(t) f(t, c(1 - p(\sigma(t))))\,dt \leq \frac{(1-p)\,d}{4}.$$

We denote by Y the set of functions $y \in C[T_0, \infty)$ satisfying

$$(1-p)\,d \leq y(t) \leq 2d \text{ on } [T, \infty) \text{ and } y(t) = y(T) \text{ on } [T_0, T]$$

and define the operator $F : Y \to C[T_0, \infty)$ by

$$(Fy)(t) = \begin{cases} (1+p)\,d - p(t)y(h(t)) + \int_T^t \int_{s_3}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty}\int_{s_1}^{\infty} f(s, y(\sigma(s)))\, ds ds_1 ds_2 ds_3, \ t \geq T \\ (Fy)(T), \quad T_0 \leq t \leq T. \end{cases}$$

By the Schauder-Tychonoff fixed point theorem F has a fixed point $y \in Y$ which is a nonoscillatory solution of (E) on $[T_0, \infty)$. Since $Z'(t) = \int_t^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty}\int_{s_1}^{\infty} f(s, y(\sigma(s)))ds ds_1 ds_2 > 0$, it follows that $\lim_{t \to \infty} Z(t) = \beta \in [(1-p)\,d, 2d]$. This completes the proof.

Now we find conditions under which the conditions of Theorem 2.1 (respectively Theorem 2.2) yields

$$\lim_{t \to \infty} \frac{y(t)}{R(t)} = \text{constant} \neq 0 \ (\text{respectively} \lim_{t \to \infty} y(t) = \text{constant} \neq 0). \qquad \dots (13)$$

Lemma 2.3 — Let $y(t)$ be a nonoscillatory solution of (E) such that $\lim_{t \to \infty} Z(t) = c \neq 0$. If

$$\lim_{t \to \infty} p(t) = p_0 \in [0, 1] \qquad \dots (14)$$

then
$$\lim_{t \to \infty} y(t) = \frac{c}{1+p_0}.$$

Lemma 2.4 — Let $y(t)$ be a nonoscillatory solution of (E) such that $\lim_{t \to \infty} \frac{Z(t)}{R(t)} = c \neq 0$. If

$$\lim_{t \to \infty} p(t) \frac{R(h(t))}{R(t)} = q_0 \in [0, 1) \qquad \dots (15)$$

then
$$\lim_{t \to \infty} \frac{y(t)}{R(t)} = \frac{c}{1+q_0}.$$

The proofs of Lemma 2.3 and 2.4 can be modelled as that of in [10] and hence the details are omitted.

Combining Theorems 2.1 and 2.2 with Lemmas 2.4 and 2.3 we have the following theorems, which ensure the existence of nonoscillatory solution of (E) having the property (13).

Theorem 2.5 — *Assume that (15) holds. Then eq. (E) has a nonoscialltory solution y(t) such that* $\lim_{t \to \infty} \frac{y(t)}{R(t)}$ *= constant* $\neq 0$, *if and only if (5) is satisfied.*

Theorem 2.6 — *Assume that (14) holds. Then eq. (E) has a nonoscillatory solution y (t) such that* $\lim_{t \to \infty} y(t)$ *= constant* $\neq 0$, *if and only if (9) is satisfied.*

Example 2.1 — Consider the neutral differential equation

$$\left(t^2\left(y(t)+\frac{t-\tau}{2(t-\tau-1)}y(t-\tau)\right)''\right)''+\frac{4(t-l)^3}{t^3(t-l-1)^3}y^3(t-l)=0 \qquad \text{... (16)}$$

where $t \geq \max\{\tau + l + 1, \tau + 2\}$, τ and l are positive constants. It is easy to check that condition (9) is satisfied but condition (5) is not satisfied for the eq. (16). Therefore, by Theorems 3.1 and 3.5 there exists no nonoscillatory solution of (16) such that $\lim_{t \to \infty} \frac{y(t)}{R(t)}$ = constant $\neq 0$, and by Theorem 3.2 and 3.6, there exists a nonoscillatory solution of (16) such that $\lim_{t \to \infty} y(t)$ = constant $\neq 0$. In fact $y(t) = \frac{t-1}{t}$ is such a solution of eq. (16).

3.3 Oscillation Results

In this section we establish conditions for the oscillation of all solutions of eq. (*E*). We begin with the following definition :

Definition 3.1 — Eq. (*E*) is called strongly superlinear if there exists a number $\alpha > 1$ such that

$$\frac{|f(t, y_1)|}{|y_1|^{\alpha}} \geq \frac{|f(t, y_2)|}{|y_2|^{\alpha}} \text{ for } |y_1| > |y_2|, y_1 y_2 > 0.$$

Eq. (*E*) is called strongly sublinear if there exists a number $\beta < 1$ such that

$$\frac{|f(t, y_1)|}{|y_1|^{\beta}} \leq \frac{|f(t, y_2)|}{|y_2|^{\beta}} \text{ for } |y_1| > |y_2|, y_1 y_2 > 0.$$

Theorem 3.2 — *Let f be strongly sublinear and $\sigma(t) \leq t$. A necessary and sufficient condition for all solutions of eq. (E) are oscillatory is that*

$$\int_{t_0}^{\infty} |f(t, c(1-p(\sigma(t)))R(t))|\,dt = \infty \qquad \text{... (17)}$$

for all $c \neq 0$.

Proof : The necessary of condition (17) follows from the sufficiency part of Theorem 2.1. Now we prove the sufficiency of condition (17). Assume that there exist a nonoscillatory solution $y(t)$ of (*E*). Without loss of generality we may assume that $y(t)$ is eventually positive. From Lemmas 1.1, 1.3 and 1.5 there exist $T \geq t_0$ and $k > 0$ such that

$$Z(t) > 0, Z'(t) > 0 \text{ and } (r(t)Z''(t))' > 0 \text{ for } t \geq T,$$

$$y(\sigma(t)) \geq (1-p(\sigma(t)))Z(\sigma(t)) \text{ for } t \geq T \qquad \text{... (18)}$$

and

$$\frac{1}{2}(r(t)Z''(t))'R(t) \leq Z(t) \leq kR(t), t \geq T. \qquad \text{... (19)}$$

Since $\sigma(t) \leq t$ and $(r(t) Z''(t))'' < 0$, we have from Lemma 1.3,

$$Z(\sigma(t)) \geq \frac{1}{2}(r(t) Z''(t))' R(\sigma(t)), t \geq T. \quad \text{... (20)}$$

From (18)-(20) and the strong sublinearity of f, we have

$$[-((r(t) Z''(t))')^{1-\beta}]' = (1-\beta)\ ((r(t) Z''(t))')^{-\beta} f(t, y(\sigma(t)))$$

$$\geq \frac{1-\beta}{(2k)^{\beta}} f(t, k(1-p(\sigma(t))) R(\sigma(t))).$$

Integrating the last inequality from T to t, we obtain

$$\frac{1-\beta}{(2k)^{\beta}} \int_T^t f(s, k(1-p(\sigma(s))) R(\sigma(s)))\, ds \leq [(rT) Z''(t))']^{1-\beta}$$

which leads to $\int_T^{\infty} f(s, k(1-p(\sigma(s))) R(\sigma(s)))\, ds < \infty$

and hence contradicts (17). This completes the proof.

Theorem 3.3 — *Let f be strongly superlinear and $\sigma(t) \geq t$. Then a necessary and sufficient condition for all solutions of eq. (E) are oscillatory is that*

$$\int_{t_0}^{\infty} R(t)\, | f(t, c(1-p(\sigma(t))))|\, dt = \infty \quad \text{... (21)}$$

for all $c \neq 0$.

PROOF : The necessity of condition (21) follows from Theorem 2.2. We prove the sufficiency of condition (21). Assume that there exists a nonoscillatory solution $y(t)$ of (E). Without loss of generality we may assume that $y(t)$ is eventually positive. From Lemmas 1.1, 1.4 and 1.5 there exists $T_1 \geq t_0$ such that

$$Z(t) > 0, Z'(t) > 0, (r(t) Z''(t))' > 0, (r(t) Z''(t))'' < 0, t \geq T_1,$$

$$y(\sigma(t)) \geq (1-p(\sigma(t))) Z(\sigma(t))$$

and

$$Z(t) \geq \int_T^t R(s, T) f(s, y(\sigma(s)))\, ds, t \geq T \geq T_1. \quad \text{... (22)}$$

We have from $Z(t) > 0, Z'(t) > 0$, there exists a $k > 0$ such that $Z(t) \geq k$ for $t \geq T$ and hence $y(t) \geq k(1-p(t))$ for $t \geq T$. From the strong superlinearity and $\sigma(t) \geq t$, we have

$$f(t, y(\sigma(t))) \geq \frac{f(t, k(1-p(\sigma(t)))}{(k(1-p(\sigma(t)))^{\alpha}} ((1-p(\sigma(t))) Z(\sigma(t)))^{\alpha}$$

$$\geq k^{-\alpha} Z^{\alpha}(t) f(t, k(1-p(\sigma(t)))). \quad \text{... (23)}$$

From (22) and (23) it follows that

$$Z(t) \geq \int_T^t k^{-\alpha} Z^{\alpha}(s)\, R(s, T) f(s, k(1-p(\sigma(s))))\, ds$$

and hence $$\left(\left[\int_T^t k^{-\alpha} Z^{\alpha}(s)\, R(s, T) f(s, k(1-p(\sigma(s))))\, ds\right]^{1-\alpha}\right)'$$

$$= \frac{(1-\alpha) k^{-\alpha} Z^{\alpha}(t)\, R(t, T) f(t, k(1-p(\sigma(t))))}{\left[\int_T^t k^{-\alpha} Z^{\alpha}(s)\, R(s, T) f(s, k(1-p(\sigma(s))))\, ds\right]^{\alpha}}$$

$$\geq (1-\alpha) Z^{-\alpha}(t)\, k^{-\alpha} Z^{\alpha}(t)\, R(t, T) f(t\, k(1-p(\sigma(t))))$$

$$= (1-\alpha) k^{-\alpha} R(t, T) f(t, k(1-p(\sigma(t))))$$

which implies that

$$R(t, T) f(t, k(1-p(\sigma(t)))) \leq \frac{k^{\alpha}}{1-\alpha}$$

$$\left[\left(\int_T^t k^{-\alpha} Z^{\alpha}(s)\, R(s, T) f(s, k(1-p(\sigma(s))))ds\right)^{1-\alpha}\right]'.$$

Integrating the last inequality from T_2 to t and using $\alpha > 1$, we obtain

$$\int_{T_2}^t R(s, T) f(s, k(1-p(\sigma(s))))\, ds \leq \frac{k^{\alpha}}{\alpha-1}\left[\int_T^{T_2} k^{-\alpha} Z^{\alpha}(s)\, R(s, T) f(s, k(1-p(\sigma(s))))ds\right]^{1-\alpha}.$$

Hence $\int_{T_2}^{\infty} R(t) f(t, k(1-p(\sigma(t))))\, dt < \infty$, a contradiction to (21). The proof is complete.

Theorem 3.4 — *Assume that there exists a continuous function $q(t)$ such that*

$$\frac{f(t, u)}{u} \geq Mq(t) > 0 \text{ for all } u \neq 0, t \geq t_0 \qquad \text{... (24)}$$

and $$\sigma(t) \leq t \text{ and } \sigma'(t) \geq c > 0 \text{ for } t \geq t_0. \qquad \text{... (25)}$$

If there exists a positive differentiable function $\rho(t)$ such that

$$\lim_{t \to \infty} \sup \int_{t_0}^t \rho(s)\left[(1-p(\sigma(s)))\, q(s) - \frac{(\rho'(s))^2}{2cMR'(s)\, \rho^2(s)}\right] ds = \infty \qquad \text{... (26)}$$

then all solutions of eq. (E) are oscillatory.

PROOF : Let $y(t)$ be a nonoscillatory solution of (E) and assume without loss of generality that $y(t)$ is eventually positive. From Lemmas 1.1 and 1.5, we have

$$Z(t) > 0, Z(\sigma(t)) > 0, Z'(t) > 0 \text{ and } (r(t) Z''(t))' > 0 \text{ for } t \geq T$$

and $$y(\sigma(t)) \geq (1 - p(\sigma(t))) Z(\sigma(t)).$$

Define $$\omega(t) = \frac{\rho(t)(r(t) Z''(t))'}{Z(\sigma(t))}, t \geq T.$$ Then in view of (4), (24) and (25), we have

$$\omega'(t) \leq -M(1 - p(\sigma(t))) \rho(t) q(t) + \frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{cR'(t) \omega^2(t)}{2\rho(t)}, t \geq T$$

$$\leq -M(1 - p(\sigma(t))) \rho(t) q(t) + \frac{(\rho'(t))^2}{2c\rho(t) R'(t)}.$$

Integrating the last inequality from T to $t \geq T$, we obtain

$$\int_T^t \rho(s) \left[1 - p(\sigma(s))) q(s) - \frac{(\rho'(s))^2}{2cM\rho^2(s) R'(s)} \right] ds \leq \frac{\omega(T)}{M}$$

and this contradicts (26). Thus the proof is complete.

For the linear equation

$$(y(t) + p(t) y(t - \tau))^{(iv)} + q(t) y(t - \sigma) = 0. \quad \ldots (27)$$

where τ and σ are non-negative constants, we obtain from Theorem 3.4 the following :

Corollary 3.5 — Suppose that $q(t) \geq 0$ for all $t \geq t_0$ and there exists a positive differentiable function $\rho(t)$ such that

$$\limsup_{t \to \infty} \int_{t_0}^t \rho(s) \left[(1 - p(t - \sigma)) q(s) - \left(\frac{\rho'(s)}{s\rho(s)} \right)^2 \right] ds = \infty,$$ then all solutions of eq. (27) are oscillatory.

We conclude this paper with the following examples.

Example 3.1 — Consider the neutral differential equation

$$\left(t^2 (y(t) + \frac{1}{\sqrt{t-1}} y(t-1))'' \right)'' ty^{\frac{1}{3}}(t-1) = 0, t \geq 3. \quad \ldots (28)$$

It is easy to see that all conditions of Theorem 3.2 are satisfied and hence all solutions of (28) are oscillatory.

Example 3.2 — Theorem 3.3 implies that all solutions of the neutral differential equation

$$\left(\frac{1}{t+\pi} \left(y(t) + \frac{1}{t+\pi} y(t-\pi) \right)'' \right)'' + t^2 y^3(t+\pi) = 0, t \geq \pi$$

are oscillatory.

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