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ON THE LIMITATIONS OF THE LINEAR GROWTH RATE IN VERONIS' AND STERN'S THERMOHALINE CONFIGURATIONS

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Modified upper limits for the complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable for Veronis'[1] and Stern's[2] thermohaline configurations are derived which in particular yield sufficient conditions for the validity of 'principle of exchange of stabilities' for these configurations. The modified results improve upon the results of Banerjee *et al.*[3] and those of Gupta *et al.*[4] in this direction and are uniformly valid for all combinations of dynamically free or rigid boundaries.

Key Words : Thermohaline Convection; Rayleigh Numbers; Lewis Number; Prandtl Number

1. INTRODUCTION

The hydrodynamic instability that manifests under appropriate conditions in a static horizontal initially homogeneous viscous, Boussinesq liquid layer of infinite horizontal extension and finite vertical depth, which is kept under the simultaneous action of a uniform vertical temperature gradient and a vertical concentration gradient opposing gravity is known as thermohaline instability. Two fundamental configurations have been studied in this context, the first one of Stern's[2] wherein the temperature gradient is stabilizing while the concentration gradient is destabilizing and the second one Veronis'[1] wherein the temperature gradient is destabilizing while the concentration gradient is stabilizing. The problem of derivation of upper limits for the complex growth rate of an arbitrary motion of growing amplitude in thermohaline problems is an important problem especially when both the boundaries are not dynamically free so that exact solutions in closed form are not obtainable and one has to depend upon numerical solutions which are rather laborious. In this situation, derivation of certain integral estimates acquire great importance, for, they enable us to obtain sufficient conditions of stability and define a possible range of parameters for growing perturbations in case of instability.

Banerjee *et al.*[3] formulated a novel way of combining the governing equations and boundary conditions for each of the Veronis'[1] and Stern's[2] thermohaline configuration and derived a semi-circle theorem prescribing upper limits for the complex growth rate of an arbitrary oscillatory perturbation neutral or unstable. Gupta *et al.*[4] derived sufficient conditions for the validity of principle of exchange of stability (PES) in Veronis' and Stern's thermohaline configurations. However, a close and critical examination of the proofs of the results of Banerjee *et al.*[3] and Gupta *et al.*[4] yield the following drawbacks in their analysis.

(*i*) Many positive definite integrals have been deleted in deriving the final results.

(*ii*) Banerjee *et al.*[3] have obtained an upper estimate for the complex growth rate in an appropriate combination with the thermohaline Rayleigh number and Prandtl number. However, the independence of this estimate from the Lewis number raises τ doubts about the accuracy of the concerned bound and it appears that a more rigid limitation than Banerjee *et al.*[3] can be obtained on the concerned growth rate by the inclusion of τ .

(*iii*) Gupta *et al.*[4] have deleted the positive definite integrals and have obtained an upper estimate for the term involving the concentration in such a way that the final inequality involves only the parameters of problem but not the complex growth rate and the wave number thereby precluding the possibility of obtaining an upper estimate for growth rate.

Keeping in view the above inherent drawbacks in the analysis of Banerjee *et al.*[3] and Gupta *et al.*[4] one strongly feels that these weaknesses are superficial and can be remedied by taking into account most of the deleted positive definite integrals in their analysis and as a consequence, improved upper limits for the complex growth rate of an arbitrary oscillatory perturbation of growing amplitude can be prescribed which in particular also yields sufficient condition for characterizing the non-oscillatory motions of these problems. The present paper is precisely in this direction.

2. Mathematical Formulation and Analysis

The relevant governing equations and boundary conditions of thermohaline instability are given by[3]

$$(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right)\omega = R_T a^2\,\theta - R_s a^2\,\phi, \qquad \dots (1)$$

$$(D^2 - a^2 - p)\,\theta = -\,\omega, \qquad \dots (2)$$

$$\{\tau(D^2 - a^2) - p\}\ \phi = -\,\omega, \qquad \dots (3)$$

with

$$\omega = \theta = \phi = 0 = D^2\,\omega \text{ at } z = 0 \text{ and } z = 1, \qquad \dots (4)$$

or

$$\omega = \theta = \phi = 0 = D\,\omega \text{ at } z = 0 \text{ and } z = 1, \qquad \dots (5)$$

or

$$\left.\begin{array}{l}\omega = \theta = \phi = 0 = D\,\omega \text{ at } z = 0\\ \text{and}\\ \omega = \theta = \phi = 0 = D^2\,\omega \text{ at } z = 1\end{array}\right\}, \qquad \dots (6)$$

or

$$\left.\begin{array}{l}\omega = \phi = \theta = 0 = D\,\omega \text{ at } z = 0\\ \text{and}\\ \omega = \theta = \phi = 0 = D\,\omega \quad \text{at } z = 1\end{array}\right\}, \qquad \dots (7)$$

where z is real independent variable such that $0 \le z \le 1, D = \frac{d}{dz}$, a^2 is a constant, $\sigma > 0$ is a constant, $\tau > 0$ is a constant, R_T and R_S are positive constants for Veronis' configuration and negative constants for Stern's configuration, $p = p_r + ip_i$ is a complex constant in general such that p_r and p_i are real constants and as a consequence the dependent variables

$$\omega(z) = \omega_r(z) + i\,\omega_i(z),\ \theta(z) = \theta_r(z) + i\,\theta_i(z)$$

and $\phi(z) = \phi_r(z) + i\,\phi_i(z)$ are complex valued functions of the real variable z such that $\omega_r(z)$, $\omega_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_r(z)$ and $\phi_i(z)$ are real valued functions of the real variable z. The meaning of the symbols from physical point of view are as follows : z is the vertical coordinate, $\frac{d}{dz}$ is differentiation along the vertical direction, a^2 the square of wave number, σ the Prandtl number, τ the Lewis number, R_T the thermal Rayleigh number, R_S the concentration Rayleigh number, p the complex growth rate, ω the vertical velocity, θ the temperature and ϕ the concentration. It may be further noted that eqs. (1)-(7) describe an eigenvalue problem for p and govern thermohaline instability for any combination of dynamically free and rigid boundaries.

We prove the following theorems :

Theorem — *If* $(p, \omega, \theta, \phi), p = pr + ipi, p_r \geq 0, p_i \neq 0$ *is a solution of eqs. (1)-(3) with either of the boundary conditions (4)-(7) and* $R_T > 0, R_S > 0$ *then*

$$|p|^2 < \frac{R_S\,\sigma - \frac{27}{4}\tau^2\,\Pi^4}{\frac{\tau^2\,\pi^4}{R_S^2\,\sigma^2}\,(6R_S\,\sigma + 13\,\tau^2\,\Pi^4)}. \qquad \text{... (8)}$$

PROOF : Multiplying eq. (1)-(3) by $\omega^*, -R_T a^2\,\theta^*$ and $-R_S a^2\,\phi^*$ respectively, integrating over the vertical range of z and adding the equations so obtained we get

$$\int_0^1 \omega^*\,(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right)\omega\,dz - R_T a^2 \int_0^1 \theta^*\,(D^2 - a^2 - p)$$

$$\theta\,dz + R_S a^2 \int_0^1 \phi^*\,(\tau\langle D^2 - a^2\rangle - p)\,\phi\,dz$$

$$= 2\,R_T a^2\,Re\left(\int_0^1 \theta\,\omega^*\,dz\right) - 2\,R_S a^2\,Re\left(\int_0^1 \phi\,\omega^*\,dz\right), \qquad \text{... (9)}$$

where '*' denotes complex conjugation and Re denotes the real part.

Integrating the left hand side of eq. (9) an appropriate number of times, using either of the boundary conditions (4)-(7), substituting the result in eq. (9) and then equating the imaginary parts yields for $p_i \neq 0$, the following equation :

$$\frac{1}{\sigma}\int_0^1 |D\,\omega|^2\,dz + \frac{a^2}{\sigma}\int_0^1 |\omega|^2\,dz = -R_T a^2 \int_0^1 |\theta|^2\,dz + R_S a^2 \int_0^1 |\phi|^2\,dz. \qquad \text{... (10)}$$

Further multiplying each of eqs. (2) and (3) by their complex conjugates and integrating by parts over the vertical range of z an appropriate number of times an making use of either of boundary conditions (4)-(7), we get

$$I_1^2 + |p|^2 \int_0^1 |\theta|^2 dz = \int_0^1 |\omega|^2 dz, \qquad \text{... (11)}$$

and

$$I_2^2 + \frac{|p|^2}{\tau^2} \int_0^1 |\phi|^2 dz = \frac{1}{\tau^2} \int_0^1 |\omega|^2 dz, \qquad \text{... (12)}$$

where

$$I_1^2 = \int_0^1 (|D^2\theta|^2 + 2a^2|D\theta|^2 + a^4|\theta|^2)\, dz + 2pr \int_0^1 (|D\theta|^2 + a^2|\theta|^2)\, dz \qquad \text{... (13)}$$

and

$$I_2^2 = \int_0^1 (|D^2\phi|^2 + a^4|\phi|^2 + 2a^2|D\phi|^2)\, dz + 2pr \int_0^1 (|D\phi|^2 + a^2|\phi|^2)\, dz. \qquad \text{... (14)}$$

We note that since $pr \geq 0$, I_1^2 and I_2^2 are positive definite. Combining eqs. (10), (11) and (12), we have

$$\frac{1}{\sigma} \int_0^1 |D\omega|^2 dz + \frac{a^2}{\sigma} \int_0^1 |\omega|^2 dz = -\frac{R_T a^2}{|p|^2} \int_0^1 |\omega|^2 dz$$

$$+ \frac{R_S a^2}{|p|^2} \int_0^1 |\omega|^2 dz + \frac{R_T a^2 I_1^2}{|p|^2} - \frac{R_S a^2 \tau^2 I_2^2}{|p|^2}. \qquad \text{... (15)}$$

Now it follows from eqs. (10), (11), (12) and (15) that

$$a^2 \int_0^1 |\phi|^2 dz > \frac{1}{R_S \sigma} \int_0^1 (|D\omega|^2 + a^2|\omega|^2)\, dz, \qquad \text{... (16)}$$

$$I_1^2 < \int_0^1 |\omega|^2 dz, \qquad \text{... (17)}$$

$$\int_0^1 |\omega|^2 dz > \tau^2 \int_0^1 |D^2\phi|^2 dz + 2a^2\tau^2 \int_0^1 |D\phi|^2 dz + a^4\tau^2 \int_0^1 |\phi|^2 dz \qquad \text{... (18)}$$

$$\int_0^1 |\omega|^2 dz > |p|^2 \int_0^1 |\phi|^2 dz, \qquad \text{... (19)}$$

$$I_2^2 > \int_0^1 |D^2\phi|^2 dz + 2a^2 \int_0^1 |D\phi|^2 dz + a^4 \int_0^1 |\phi|^2 dz. \qquad \text{... (20)}$$

Since $\omega(0) = 0 = \omega(1)$ and $\phi(0) = 0 = \phi(1)$, ... (21)

therefore by Rayleigh-Ritz inequality (Schultz[5]), we have

$$\int_0^1 |D\,\omega|^2\,dz \geq \Pi^2 \int_0^1 |\omega|^2\,dz, \qquad \text{... (22)}$$

$$\int_0^1 |D\,\phi|^2\,dz \geq \Pi^2 \int_0^1 |\phi|^2\,dz. \qquad \text{... (23)}$$

Further

$$\int_0^1 |D^2\,\phi|^2\,dz = -\int_0^1 \phi^*\,D\,\phi\,dz = +\left|-\int_0^1 \phi^*\,D^2\,\phi\,dz\right| \leq \int_0^1 |\phi|\,|D^2\,\phi|\,dz,$$

which upon utilizing (23) gives

$$\Pi^2 \int_0^1 |\phi|^2\,dz \leq \left\{\int_0^1 |\phi|^2\,dz\right\}^{1/2} \left\{\int_0^1 |D^2\,\phi|^2\,dz\right\}^{1/2} \quad \text{(Schwartz inequality)}$$

or

$$\int_0^1 |D^2\,\phi|^2\,dz \geq \Pi^4 \int_0^1 |\phi|^2\,dz. \qquad \text{... (24)}$$

Using inequality (16)-(24) appropriately, we get

$$I_2^2 > \frac{27\,\pi^4}{4\,R_S\,\sigma} \int_0^1 |\omega|^2\,dz \qquad \text{... (25)}$$

$$\frac{1}{\sigma} \int_0^1 |D\,\omega|^2\,dz \geq \frac{10\,\tau^4\,\pi^8\,a^2}{R_S^2\,\sigma^3} \int_0^1 |\omega|^2\,dz + \frac{3\,\tau^2\,\pi^4\,a^2}{R_S\,\sigma^2} \int_0^1 |\omega|^2\,dz \qquad \text{... (26)}$$

and

$$\frac{a^2}{\sigma} \int_0^1 |\omega|^2\,dz \geq \frac{3\,\tau^2\,\pi^8}{R_S^2\,\sigma^3}\,a^2 \int_0^1 |\omega|^2\,dz + \frac{3\,\tau^2\,\pi^4\,a^2}{R_S\,\sigma^2} \int_0^1 |\omega|^2\,dz. \qquad \text{... (27)}$$

It now follows from eq. (15) and inequalities (17), (25), (26) and (27) that

$$\frac{a^2}{\sigma} \int_0^1 \left\langle \left\{\frac{\tau^2\,\pi^4}{R_S^2\,\sigma^2}\,(6R_S\,\sigma + 13\,\tau^2\,\Pi^4)\right\} - \left\{\frac{R_S\,\sigma - \frac{27}{4}\,\tau^2\,\Pi^4}{|p|^2}\right\} \right\rangle |\omega|^2\,dz < 0 \qquad \text{... (28)}$$

which clearly implies that

$$|p|^2 < \frac{R_S\sigma - \frac{27}{4}\Pi^4\tau^2}{\frac{\tau^2\Pi^4}{R_S^2\sigma^2}(6R_S\sigma + 13\tau^2\Pi^4)}.$$

This completes the proof of the theorem.

Theorem 1 implies that the complex growth rate of an arbitrary oscillatory motion of growing amplitude, in the thermohaline instability of Veronis' type, lies inside a semicircle in upper half of $p_r - p_i$ plane, whose centre is at origin and radius is

$$\left\{\frac{R_S\sigma - \frac{27}{4}\tau^2\pi^4}{\frac{\tau^2\pi^4}{R_S^2\sigma^2}(6R_s\sigma + 13\tau^2\pi^4)}\right\}^{1/2} \quad \text{... (29)}$$

Corollary 1 — If $(p, \omega, \theta, \phi), p = pr + ipi, pr \geq 0, pi \neq 0$ is a solution of eqs. (1)-(3) with either of the boundary conditions (4)-(7) with $R_T > 0, R_S > 0$ and if $\frac{R_S\sigma}{\frac{27}{4}\tau^2\pi^4} \leq 1$

or $$R_S\sigma \leq \frac{27}{4}\tau^2\pi^4 \text{ then } pi = 0 \quad \text{... (30)}$$

PROOF : Follows from the Theorem 1.

Corollary 1 implies that for thermohaline instability of Veronis' type if $\frac{R_S\sigma}{27/4\,\tau^2\pi^4} \leq 1$ then an arbitrary neutral or unstable mode is definitely non-oscillatory in character and in particular PES is valid.

***Theorem* 2** — *If $(p, \omega, \theta, \phi), p = pr + ipi, pr \geq 0, pi \neq 0$ is a solution of eqs. (1)-(3) with either of the boundary conditions (4)-(7) and $R_T < 0, R_S < 0$ then,*

$$|p|^2 < \frac{|R_T|\sigma - \frac{27}{4}\Pi^4}{\frac{\Pi^4}{|R_T|^2\sigma^2}(6|R_T|\sigma + 13\pi^4)}. \quad \text{... (31)}$$

PROOF : Putting $R_T = -|R_T|$ and $R_S = -|R_S|$ eqs. (10) and (12) in the present case assume the following forms :

$$\frac{1}{\sigma}\int_0^1 |D\omega|^2 dz + \frac{a^2}{\sigma}\int_0^1 |\omega|^2$$

$$= |R_T| a^2 \int_0^1 |\theta|^2 dz - |R_S| a^2 \int_0^1 |\phi|^2 dz, \qquad \text{... (32)}$$

$$\frac{1}{\sigma} \int_0^1 |D\omega|^2 dz + \frac{a^2}{\sigma} \int_0^1 |\omega|^2 dz$$

$$= \frac{|R_T| a^2}{|p|^2} \int_0^1 |\omega^2| dz - \frac{|R_S| a^2}{|p|^2} \int_0^1 |\omega|^2 dz + \frac{|R_T|}{|p|^2} a^2 I_1^2 + \frac{|R_S|}{|p|^2} a^2 I_2^2. \qquad \text{... (33)}$$

Now it follows from eqs. (32), (12), (13) and (33) that

$$a^2 \int_0^1 |\theta|^2 dz > \frac{1}{|R_T| \sigma} \int_0^1 (|D\omega|^2 + a^2 |\omega|^2) dz \qquad \text{... (34)}$$

$$I_2^2 > \frac{1}{\tau^2} \int_0^1 |\omega|^2 dz, \qquad \text{... (35)}$$

$$\int_0^1 |\omega|^2 dz > \int_0^1 (|D^2\theta|^2 + 2a^2 |D\theta|^2 + a^4 |\theta|^2) dz, \qquad \text{... (36)}$$

$$\int_0^1 |\omega|^2 dz > |p|^2 \int_0^1 |\theta|^2 dz, \qquad \text{... (37)}$$

$$I_1^2 > \int_0^1 |D^2\theta|^2 dz + 2a^2 \int_0^1 |D\theta|^2 dz + a^4 \int_0^1 |\theta|^2 dz, \qquad \text{... (38)}$$

since $\omega(0) = 0 = \omega(1)$ and $\theta(0) = 0 = \theta(1)$. ... (39)

Therefore, by Rayleigh-Ritz inequality (Schultz[5]), we have

$$\int_0^1 |D\omega|^2 dz \geq \Pi^2 \int_0^1 |\omega|^2 dz, \qquad \text{... (40)}$$

$$\int_0^1 |D\theta|^2 dz \geq \Pi^2 \int_0^1 |\theta|^2 dz, \qquad \text{... (41)}$$

and

$$\int_0^1 |D^2\theta|^2 dz \geq \Pi^4 \int_0^1 |\theta|^2 dz. \qquad \text{... (42)}$$

Using inequality (34)-(42) appropriately, we have

$$p_1^2 > \frac{27\pi^4}{4|R_T|\sigma}\int_0^1 |\omega|^2\,dz, \qquad \text{... (43)}$$

$$\frac{1}{\sigma}\int_0^1 |D\omega|^2\,dz \geq \frac{10\pi^4 a^2}{|R_T|^2\sigma^3}\int_0^1 |\omega|^2\,dz + \frac{3\pi^4 a^2}{|R_T|\sigma^2}\int_0^1 |\omega|^2\,dz, \qquad \text{... (44)}$$

and

$$\frac{a^2}{\sigma}\int_0^1 |\omega|^2\,dz \geq \frac{3\pi^4 a^2}{|R_T|^2\sigma^3}\int_0^1 |\omega|^2\,dz + \frac{3\pi^4 a^2}{|R_T|\sigma^2}\int_0^1 |\omega|^2\,dz. \qquad \text{... (45)}$$

It now follows from eq. (33) and inequalities (35), (43), (44) and (45) that

$$\frac{a^2}{\sigma}\int_0^1 \left\langle \left\{\frac{\pi^4}{|R_T|^2\sigma^2}(6|R_T|\sigma dt + 13\pi^4)\right\} - \left\{\frac{|R_T|\sigma - \frac{27}{4}\pi^4}{|p|^2}\right\}\right\rangle \int_0^1 |\omega|^2 < 0 \qquad \text{... (46)}$$

which clearly implies that

$$|p|^2 < \frac{|R_T|\sigma - \frac{27}{4}\pi^4}{\frac{\pi^4}{|R_T|^2\sigma^2}(6|R_T|\sigma + 13\pi^4)}. \qquad \text{... (47)}$$

This completes the proof of the theorem.

Corollary 2 — If $(p, \omega, \theta, \phi), p = pr + ipi, pr \geq 0, pi \neq 0$ is a solution of eqs. (1)-(3) with either of the boundary conditions (4)-(7) with $R_T < 0, R_S < 0$ and if

$$\frac{|R_T|\sigma}{\frac{27}{4}\pi^4} \leq 1 \text{ then } pi = 0 \qquad \text{... (48)}$$

PROOF : Follows from Theorem 2.

The essential contents of Theorem 2, and Corollary 2 are similar to that of the Theorem 1 and Corollary 1 respectively.

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