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LOCAL AUTOMORPHISMS OF NEST ALGEBRAS*

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We prove that every local automorphism of the algebra of all $n \times n$ upper triangular matrices is an automorphism, and locally generalize this to nest algebras and prove that every continuous in the strong operator topology local inner automorphism of a nest algebra acting on a reflexive Banach space is an inner automorphism. In addition, we prove that every strongly continuous local automorphism of the nest algebra is an automorphism if $0_+ \neq 0$ or $X_- \neq X$ in the nest $\mathcal{N}$.

Key Words : Nest Algebra; Local Automorphism

1. INTRODUCTION AND PRELIMINARIES

In this paper we study the local automorphisms of nest algebras on Banach spaces. A linear mapping θ of an algebra $\mathcal{A}$ onto itself is called a local automorphism if for each $a \in \mathcal{A}$, there exists an automorphism θ_a of $\mathcal{A}$ such that $\theta(a) = \theta_a(a)$. This notion was introduced by Larson and Sourour in[7]. They proved that every surjective local automorphism of $B(X)$, the algebra of all bounded linear operators on a complex infinite dimensional Banach space X, is an automorphism. In[1], Brevšar and Švemrl generalized this to real Banach spaces; and in[2], they also obtained the same result on an infinite dimensional separable Hilbert space by removing the assumption of surjectivity. In the finite dimensional case, the situation is somewhat different. In[7], the authors proved that every local automorphism of $\mathcal{M}_n$, the algebra of all $n \times n$ complex matrices, is an automorphism or antiautomorphism. The following question seems natural: Is every local automorphism of a nest algebra on a Banach space an automorphism? We prove that every local automorphism of $\mathcal{T}_n$, the algebra of all $n \times n$ upper triangular complex matrices, is an automorphism. We then prove that every continuous in the strong operator topology local inner automorphism of a nest algebra on a banach space is an inner automorphism. Furthermore, under some conditions a continuous strongly local automorphism is an automorphism. This is new even in the Hilbert space case.

We recall that Ringrose has proved that every algebraic isomorphism between nest algebras on Hilbert spaces is spatial[8]. Lance also proved this result in a different way[6]. This was generalized to Banach spaces[3].

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Throughout this paper, let X be a real or complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. The symbol "$\subset$" denotes proper inclusion of subspaces. A nest is a totally ordered lattice $\mathcal{N}$ of closed subspaces of X which contains 0 and X, and is closed under arbitrary closed linear spans (denoted by $\vee$) and intersections (denoted by $\wedge$). The nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all bounded linear operators leaving each element of $\mathcal{N}$ invariant.

Let $\mathcal{A}$ be a subalgebra of $B(X)$. A linear mapping θ of $\mathcal{A}$ onto itself is called an automorphism if θ is a bijection and, for every A, B in $\mathcal{A}$, $\theta(AB) = \theta(A)\,\theta(B)$. If there is an invertible operator T in $B(X)$ such that $\theta(A) = TAT^{-1}$, for each $A = \mathcal{A}$, then θ is called a spatial automorphism; In this case, if $T \in \mathcal{A} \cap \mathcal{A}^{-1}$, where $\mathcal{A}^{-1} = \{A^{-1} : A \in \mathcal{A}\}$, then θ is called an inner automorphism. A linear mapping θ of $\mathcal{A}$ onto itself is called a local automorphism (resp. inner automorphism) if for every $A \in \mathcal{A}$, there is an automorphism (resp. inner automorphism) θ_A such that $\theta(A) = \theta_A(A)$. From[8], if θ is a local automorphism (resp. inner automorphism) of the nest algebra $\mathcal{T}(\mathcal{N})$, then, for every $T \in \mathcal{T}(\mathcal{N})$, there exists an invertible bounded linear operator $A_T \in B(X)$ (resp. $A_T \in \mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^{-1}$), such that $\theta(T) = A_T T A_T^{-1}$.

If $N \in \mathcal{N}$, then let $N_- = \vee\ \{L \in \mathcal{N} : L \subset N\}$, $(N \neq 0)$ and $N_+ = \wedge\ \{L \in \mathcal{N} : N \subset L\}$, $(N \neq X)$. If $x \in X$ and $f \in X^*$ (the dual Banach space of X), the rank one operator $x \otimes f$ denotes $x \otimes f(z) = f(z)\,x$, for every $z \in X$. In[4], $x \otimes f \in \mathcal{T}(\mathcal{N})$ if and only if there exists $N \in \mathcal{N}$ such that $x \in N, f \in N_-^{\perp}$, where $N_-^{\perp} = \{f \in X^* : f(y) = 0\ y \in N_-\}$.

2. Local Automorphisms of $\mathcal{T}_n$

Let $\mathcal{M}_n$ be the algebra of all $n \times n$ matrices on the complex number field $\mathbb{C}$, and let $\mathcal{T}_n$ be the algebra of all $n \times n$ upper triangular matrices. Let $e_1, e_2, \ldots, e_n$ be the vectors of the standard basis of $\mathbb{C}^n$. In other words, e_i is the order n row vector with ith component 1 and all other components 0. Let $E_{ij}\,(i,j = 1, 2, \ldots, n)$ be the standard matrix units of $\mathcal{M}_n$. That is, E_{ij} is an $n \times n$ matrix whose (i, j) entry is 1 and all other entries are 0. Let A^t be the transpose of an $n \times m$ matrix A. Then $E_{ij} = e_i^t e_j$ and $e_i A e_j^t = a_{ij}$ for each $A = (a_{ij}) \in \mathcal{M}_n$.

The following proposition is a result of the similar Theorem and automorphism theorem of nest algebras[8], we can give a direct proof.

Proposition 2.1 — *Every automorphism θ of $\mathcal{T}_n$ is inner.*

Proof : From[8], θ is spatial. So there is an invertible matrix $A \in \mathcal{M}_n$ such that $\theta(T) = ATA^{-1}$, for each $T \in \mathcal{T}_n$. We claim that $A \in \mathcal{T}_n$.

Otherwise, let $A = (a_{ij})$ and $A^{-1} = (b_{ij})$. Then there exists a nonzero entry $a_{i_0 j_i}\,(i_0 > j_0)$ of A. For each k, l with $j_0 \leq k, l \leq i_0$, then $E_{j_0 k}$ is an upper triangular matrix and $e_{i_0}(AE_{j_0 k}A^{-1})\,e_l^t$ is the (i_0, l) entry of $AE_{j_0 k}A^{-1}$. Then $e_{i_0}(AE_{j_0 k}A^{-1})\,e_l^t = a_{i_0 j_0} b_{kl} = 0$, and thus $b_{kl} = 0$, for each $k\,(j_0 \leq k)$ and $l\,(l \leq i_0)$. If $j_0 = 1$, let $l = 1$. Then every entry of the first column of A^{-1} is zero, which is impossible. If $j_0 > 1$, then fix the rows $1, 2, \ldots, j_0 - 1$ of A^{-1}. For each $1 \leq \upsilon_1 < \upsilon_2 < \ldots < \upsilon_{j_0 - 1} \leq n$, we consider the order p submatrix $A_{1, 2, \ldots, j_0 - 1}^{\upsilon_1, \upsilon_2, \ldots, \upsilon_{j_0 - 1}}$ of A, which consists of the rows 1, 2, ..., $j_0 - 1$ and the columns $\upsilon_1, \upsilon_2, \ldots, \upsilon_{j_{0-1}}$. Since its algebraic complement minor is order $n - j_0 + 1$ and $n - j_0 + 1 > n - i_0 + 1$, there exists a zero column in the complement minor. By the Laplace Expansion Formula[5], the determinant of A^{-1} is 0, which is impossible.

Theorem 2.2 — *Every local automorphism of* $\mathcal{T}_n$ *is an automorphism.*

PROOF : Let θ be a local automorphism of $\mathcal{T}_n$. Then for every $T \in \mathcal{T}_n$, there exists an invertible matrix $A_T \in \mathcal{T}_n$ such that $\theta(T) = A_T T A_T^{-1}$. Hence θ preserves the idempotents and nilpotents. So $\theta(E_{ij}^2) = \theta(E_{ij})^2$ for $i \leq j$.

We complete the proof by proving $\theta(E_{ij})\, \theta(E_{kl}) = 0$, for each $i \leq j, k \leq l$ and $j \neq k$; and $\theta(E_{il}) = \theta(E_{ij})\, \theta(E_{jl})$, for each $i \leq j \leq l$. We have to consider several cases.

Case 1 — Suppose $i \leq j, k \leq l, k < j$.

Let $\theta(E_{ij}) = AE_{ij}A^{-1}$, $\theta(E_{kl}) = BE_{kl}B^{-1}$, for some $A, A^{-1}, B, B^{-1} \in \mathcal{T}_n$. Let $C = A^{-1}B = (c_{st}) \in \mathcal{T}_n$. Then

$$\theta(E_{ij})\, \theta(E_{kl}) = c_{jk} AE_{il} B^{-1} = 0.$$

Case 2 — Suppose $i \leq j \leq l$.

By case 1 and θ preserving the idempotent, one can get $\theta(E_{ii})\, \theta(E_{ll}) = \theta(E_{ii} E_{ll})$ for all i, $l = 1, 2, \ldots, n$.

Let $p = E_{ii} + E_{il}$ for $i = j < l$; and let $q = E_{ij} + E_{jj}$ for $i < j = l$. Then p and q are idempotents. By case 1, we obtain that $\theta(E_{il}) = \theta(E_{ij})\, \theta(E_{jl})$.

If $i < j < l$, let $p = E_{ij} + E_{il} + E_{jl} + E_{jj}$, then p is an idempotent. From $\theta(p) = \theta(p)^2$, by case 1 and above paragraphs, we have $\theta E_{il}) = \theta(E_{ij})\, \theta(E_{jl})$.

Case 3 — Suppose $i \leq j < k \leq l$.

If $i < j < k < l$, let $p = E_{ij} + E_{kl}$, then p is a nilpotent. Hence from $\theta(p)^2 = 0$ and by the case 1, one gets that $\theta(E_{ij})\, \theta(E_{kl}) = 0$.

Let $p = E_{ij} + E_{ii} + E_{kk}$ for $i < j < k = l$, and let $q = E_{ii} + Eu + E_{kl}$ for $i = j < k < l$. Then p and q are idempotents. By case 1 and case 2, we have $\theta(E_{ij})\, \theta(E_{kl}) = 0$.

3. LOCAL INNER AUTOMORPHISMS OF NEST ALGEBRAS

Let $\mathcal{N}$ be a nontrivial nest on X such that for each $N \in \mathcal{N}$, dim $(N) > 1$ $(N \neq 0)$ and dim $(N_-^{\perp}) < 1$ $(N_- \neq X)$. Let α be a local inner automorphism of $\mathcal{T}(\mathcal{N})$. Then α is a linear bijection and preserves the rank one oprators of $\mathcal{T}(\mathcal{N})$. And α^{-1} is a local inner automorphism. Moreover, suppose that

α and α^{-1} are continuous in the strong operator topology.

Let $$\mathcal{N}_* = \{N \in \mathcal{N} : N \neq 0, N_- \neq X\}.$$

Definition — Fixed $N \in \mathcal{N}_*$, $x \in N, f \in N_-^{\perp}$, let

$$L_x^N = \{x \otimes g : g \in N_-^{\perp}\},\ R_f^N = \{x \otimes f : x \in N\}.$$

Obviously, L_x^N and R_f^N are norm closed right ideal and left ideal of $\mathcal{T}(\mathcal{N})$, respectively, both of which consist of the rank one operators and zero operator in $\mathcal{T}(\mathcal{N})$. By the hypothesis of $\mathcal{N}_*$ for each $x \in N$ and $f \in N_-^{\perp}$, both $L_x^N \subseteq R_f^N$ and $R_f^N \subseteq L_x^N$ can not hold.

Proposition 3.1 — *Let $\mathcal{S}$ be a subspace of $\mathcal{T}(\mathcal{N})$ consisting of rank one operators and zero in $\mathcal{T}(\mathcal{N})$. Then there exists $N \in N_*$ such that $\mathcal{S} \subseteq L_x^N$ for some $x \in N$, or $\mathcal{S} \subseteq R_f^N$ for some $f \in N_-^{\perp}$.*

PROOF : Fix a rank one operator $x_0 \otimes f_0 \in \mathcal{S}$, let $N_0 = \wedge \{P \in \mathcal{N} : x_0 \in P\}$. Then $N_0 \in \mathcal{N}$ and for each rank one operator $x_0 \otimes f \in \mathcal{T}(\mathcal{N})$, we have $f \in N_{0_-}^{\perp}$, and thus $f_0 \in N_{0_-}^{\perp\perp}$.

If $x \otimes f$ and $x_0 \otimes f_0$ are linearly dependent for each $x \otimes f \in \mathcal{S}$, then $\mathcal{S} = R_{f_0}^N \cap L_{x_0}^N$, and so, there is nothing to show. Otherwise, there exists a rank one operator $x_1 \otimes f_1$ in $\mathcal{S}$ such that $x_1 \otimes f_1$ and $x_0 \otimes f_0$ are linearly independent. By the assumption of $\mathcal{S}$, $x_1 \otimes f_1 + x_0 \otimes f_0$ is rank one, So only one case holds in the following statements:

Case 1 — x_1 and x_0 are linearly dependent, while f_1 and f_0 are linearly independent.

Case 2 — x_1 and x_0 are linearly independent, while f_1 and f_0 are linearly dependent.

We consider the case 1. In this case, we will prove that $\mathcal{S} \subseteq L_{x_0}^{N_0}$.

Let $$x_1 = \lambda x_0\ (\lambda \neq 0) \text{ and let } x \otimes f \in \mathcal{S}.$$

If $x \otimes f$ and $x_0 \otimes f_0$ are linearly dependent, then $x \otimes f \in L_{x_0}^{N_0}$. If $x \otimes f$ and $x_0 \otimes f_0$ are linearly independent, then x and x_0 are linearly dependent, while f and f_0 are linearly independent. Otherwise, from hypothesis of $\mathcal{S}$, we have that x and x_0 are linearly independent, but f and f_0 are linearly dependent. Let $f = \gamma f_0$. Also since $x \otimes f + x_1 \otimes f_1 + x_0 \otimes f_0 = x \otimes \gamma f_0 + x_0 \otimes \lambda f_1 + x_0 \otimes f_0 = x \otimes \gamma f_0 + x_0 \otimes (f_0 + \lambda f_1)$is rank one, f_0 and f_1 are linearly dependent, which contradicts the hypothesis of case 1. Let $x = \kappa x_0$. Then $x \otimes f = x_0 \otimes \kappa f \in \mathcal{T}(\mathcal{N})$, from the definition of x_0, and thus $f \in (N_0)_-^{X3}$. Hence $x \otimes f \in L_{x_0}^{N_0}$. Thus $\mathcal{S} \otimes L_{x_0}^{N_0}$.

Similarly, for case 2, we can prove that there exists $N \in N_*$ such that $\mathcal{S} \subset R_f^N$ for some $f \in N_-^{\perp}$.

Remark : Let $\mathcal{S}$ be as in Proposition 3.1. Suppose that $\mathcal{S}$ is a maximal subspace of rank one operators in $\mathcal{T}(\mathcal{N})$ in the sense that $\mathcal{S}$ is not contained properly in any such a subspace of $\mathcal{T}(\mathcal{N})$. Then $\mathcal{S}=L_x^N$ or $\mathcal{S}=R_f^M$. And if $\mathcal{S}$ is a norm closed left (resp. right) ideal of $\mathcal{T}(\mathcal{N})$ consisting of rank one operators in $\mathcal{T}(\mathcal{N})$, then $\mathcal{S}=R_f^M$ (resp. $\mathcal{S}=L_x^N$). We leave them to the readers.

Lemma 3.2 — Let $N\in \mathcal{N}_*, x\in N, f\in N_-^{\perp}$. Then

(i) there exists $\varphi(x)\in N$ such that $\alpha(L_x^N)=L_{\varphi(x)}^N$ or there exists $g_x\in N_-^{\perp}$ such that $\alpha(L_x^N)=R_{gx}^N$,

(ii) there exists $\mu_f\in N$ such that $\alpha(R_f^N)=L_{\mu f}^N$ or there exists $h_f\in N_-^{\perp}$ such that $\alpha(R_f^N)=R_{hf}^N$

PROOF : Since α and α^{-1} are local inner, for each $y\otimes h\in \mathcal{T}(\mathcal{N})$, $y\in N, h\in N_-^{\perp}$, there exist $A,B\in \mathcal{T}(\mathcal{N})\cap(\mathcal{T}(\mathcal{N})^{-1})$ such that $\alpha(y\otimes h)=A(y\otimes h)A^{-1}$ and $\alpha^{-1}(y\otimes h)=B(y\otimes h)B^{-1}$. Obviously, $Ay, By\in N$ and $(A^{-1})^*h, (B^{-1})^*h\in N_-^{\perp}$.

We only prove (i).

Let $\mathcal{S}=\alpha(L_x^N)$. Then $\mathcal{S}$ is a subspace of $\mathcal{T}(\mathcal{N})$ consisting of rank one operators of $\mathcal{T}(\mathcal{N})$. From Proposition 3.1 and above argument, we have $\mathcal{S}\subseteq L_{\varphi(x)}^N$ for some $\varphi(x)\in N$, or $\mathcal{S}\subseteq R_{g(x)}^N$ for some $g_x\in N_-^{X3}$.

Suppose $\mathcal{S}=\alpha(L_x^N)\subseteq L_{\varphi(x)}^N$, for some $\varphi(x)\in N$. Then $L_x^N=\alpha^{-1}(\mathcal{S})\subseteq\alpha^{-1}(L_{\varphi(x)}^N)$. Let $\mathcal{S}_1=\alpha^{-1}(L_{\varphi(x)}^N)$. Then by the same arguments for $\mathcal{S}_1$ $\mathcal{S}_1\subseteq L_{\phi(x)}^N$ for some $\phi(x)\in N$, or $\mathcal{S}_1\subseteq R_{h_x}^N$ for some $h_x\in N_-^{\perp}$. Obviously the latter is impossible, for otherwise, $L_x^N\subseteq\mathcal{S}_1\subseteq R_{h_x}^{N\perp}$ which contradicts the hypothesis for the nest $\mathcal{N}$. So $L_x^N\subseteq\mathcal{S}_1\subseteq L_{\phi(x)}^N$. It follows that $L_x^N=L_{\phi(x)}^N=\mathcal{S}_1=\alpha^{-1}(L_{\varphi(x)}^N)$. Hence $\alpha(L_x^N)=L_{\varphi(x)}^N$.

Suppose $\mathcal{S}=\alpha(L_x^N)\subseteq R_{gx}^N$ for some $g_x\in N_-^{\perp}$. Then $L_x^N=\alpha^{-1}(\mathcal{S})\subseteq\alpha^{-1}(R_{gx}^N)$. Let $\mathcal{S}_2=\alpha^{-1}(R_{gx}^N)$. Then by the similar way to discussing $\mathcal{S}_1$, we obtain that $\mathcal{S}_2\subseteq L_{\mu_x}^N$ for some $\mu_x\in N$. So $L_x^N\subseteq\mathcal{S}_2\subseteq L_{\mu_x}^N$, which implies $L_x^N=\mathcal{S}_2=L_{\mu_x}^N$. It follows that $\alpha(L_x^N)=R_{gx}^N$.

Remark : Let $N\in\mathcal{N}_*, u, \upsilon, \in N$. Then $L_u^N\cap L_\upsilon^N=L_u^N$ when u and υ are linearly dependent; otherwise, $L_u^N\cap L_\upsilon^N=0$. Hence *dim* $(L_u^N\cap L_\upsilon^N)=$ *dim* $(N_-^{\perp})$, or 0. If $\alpha(L_u^N)=L_x^N$ and $\alpha(L_\upsilon^N)=R_f^N$, then $\alpha(L_u^N n\cap L_\upsilon^N)=L_x^N\cap R_f^N$, and thus *dim* $\alpha(L_u^N\cap L_\upsilon^N)=1$. Since α is bijective, $\alpha(L_u^N)=L_x^N$ and $\alpha(L_\upsilon^N)=R_f^N$ can not hold simultaneously. Then, for each $N\in\mathcal{N}_*$, only two cases hold;

Case 1 — For every $x \in N$, there is $\varphi(x) \in N$ such that $\alpha(L_x^N) = L_{\varphi(x)}^N$.

Case 2 — For every $x \in N$, there is $g_x \in N_-^{\perp}$ such that $\alpha(L_x^N) = R_{g(x)}^N$.

Similarly, there are only two cases of $\alpha(R_f^N)$:

Case 1'— For each $f \in N_-^{\perp}$, there is $\mu_f \in N$ such that $\alpha(R_f^N) = L_{\mu f}^N$,

Case 2' — For each $f \in N_-^{\perp}$, there is $h_f \in N_-^N$ such that $\alpha(R_f^N) = R_{hf}^N$,

In the following lemma, we prove that case 2 and case 1' cannot happen.

Lemma 3.3 — Let $N \in \mathcal{N}_*$.

(i) $\alpha(L_x^N) = L_{\varphi(x)}^N$, for each $x \in N$.

(ii) $\alpha(R_f^N) = R_{hf}^N$, for each $f \in N_-^{\perp}$.

PROOF : We only prove (i)

(1) Assume $X \neq X_-$. If Case 2 is true for X, let $N \in \mathcal{N}_*$, $N \subset X$. If $\alpha(L_y^N) = L_{\varphi(y)}^N$ for each $y \in N$, then $L_{\varphi(y)}^N = \alpha(L_y^N) \supseteq \alpha(L_y^X) = R_{g_y}^X$; if $\alpha(L_z^N) = R_{g_z}^N$ for each $z \in N$, then $R_{g_z}^N = \alpha(L_z^N) \supset \alpha(L_z^X) = R_{h_z}^X$. But these two cases for N are impossible, which contradicts the above Remark. Then only Case 1 is true for X.

Let $N \in \mathcal{N}_*$, $N \subset X$. If Case 2 is true for N, then $\alpha(L_x^N) = R_{gx}^N$ for each $x \in N$. Hence $R_{gx}^N = \alpha(L_x^N) \supseteq \alpha(L_x^X) = L_{\varphi(x)}^X$, which is impossible. Then Case 1 is true for each $N \subset X$.

(2) Assume $X = X_-$. Let $N \in \mathcal{N}_*$, $N \subset X$. If Case 2 for N is true, then let $M \in N, N \subset M \subset X$. By the similar way to the proof of (1), case 1 and case 2 for M are not true. Then for N, case 1 is true.

Lemma 3.4 — Let $N \in \mathcal{N}_*$, $x \in N$. Then there exist $\varphi(x) \in N$ and $C_x^N \in B(N_-^{\perp})$ such that $\alpha(x \otimes f) = \varphi(x) \otimes C_x^N f$, for each $f \in N_-^{\perp}$.

PROOF : From Lemma 3.3, let $\alpha(L_x^N) = L_{\varphi(x)}^N$. Then for each $f \in N_-^{\perp}$, there exists $g_f \in N_-^{\perp}$ such that $\alpha(x \otimes f) = \varphi(x) \otimes g_f$. Let $C_x^N(f) = g_f$. Then $\alpha(x \otimes f) = \varphi(x) \otimes C_x^N(f)$. Let $h \in X^*$ such that $h(\varphi(x)) = 1$. Then $C_x^N(f) = \alpha(x \otimes f)^*(h)$, and from the continuity and linearity of α, $C_x^N \in B(N_-^{\perp})$.

Lemma 3.5 — Let $N \in \mathcal{N}_*$. Then there exists $C_N \in B(N_-^{\perp})$ such that for every $x \in N$, there is $\varphi(x) \in N$ satisfying $\alpha(x \otimes f) = \varphi(x) \otimes C_N f$, for every $f \in N_-^{\perp}$.

PROOF : Fix a nonzero vector $x_0 \in N$, and let x be a nonzero vector in N.

If x and x_0 are linearly independent, then from $\alpha(L_x^N) = L_{\varphi(x)}^N$, $\alpha(L_{x_0}^N) = L_{\varphi(x_0)}^N$, we can say $\varphi(x)$ and $\varphi(x_0)$ are linearly independent. Also since $\alpha((x + x_0) \otimes f) = \varphi(x) \otimes C_x^N f + \varphi(x_0) \otimes C_{x_0}^N f$ is rank one, $C_x^N f$ and $C_{x_0}^N f$ are linearly dependent for every $f \in N_-^{\perp}$. Hence C_x^N and $C_{x_0}^N$ are linearly dependent.

Let $C_x^N = \lambda_x C_{x_0}^N$. Then $\alpha(x \otimes f) = (\lambda_x \varphi(x)) \otimes C_{x_0}^N f$. Let $C_N = C_{x_0}^N$ and replace $\lambda_x \varphi(x)$ with $\varphi(x)$. Then $\alpha(x \otimes f) = \varphi(x) \otimes C_N f$.

Lemma 3.6 — Let $N \in \mathcal{N}_*, f \in \mathcal{N}_-^{\perp}$. Then there exist $B_f^N \in B(N)$ and $h_f \in N_-^{\perp}$ such that $\alpha(x \otimes f) = B_f^N x \otimes h_f^{\perp}$, for each $x \in N$.

PROOF : From Lemma 3.3, let $\alpha(R_f^N) = R_{h_f}^N$.

For each $x \in N$, then there exists $g_x \in N$ such that $\alpha(x \otimes f) = g_x \otimes h_f$. Let $B_f^N x = g_x$. Then $\alpha(x \otimes f) = B_f^N x \otimes h_f$. Let $u \in X$ such that $h_f(u) = 1$. Then $B_f^N x = \alpha(x \otimes f)(u)$. Thus $B_f^N \in B(N)$.

Lemma 3.7 — Let $N \in \mathcal{N}_*$. Then there exists $B_N \in B(N)$ such that for each $f \in N_-^{\perp}$, there is some $h_f \in N_-^{\perp}$ satisfying $\alpha(x \otimes f) = B_N x \otimes h_f$ for every $x \in N$.

PROOF : Let $f_0 \in N_-^{\perp}$ be a fixed nonzero vector. Let $f \in N_-^{\perp}$.

If f and f_0 are linearly dependent, let $f = kf_0$. Then from Lemma 3.6, $\alpha(x \otimes f) = B_f^N x \otimes h_f = k\,\alpha(x \otimes f_0) = B_{f_0}^N x \otimes kh_{f_0}$. Hence B_f^N and $B_{f_0}^N$ are linearly dependent.

If f and f_0 are linearly independent, then from $\alpha(R_f^N) = R_{h_f}^N$ and $\alpha(R_{f_0}^N) = R_{hf_0}^N$, h_f and h_{f_0} are linearly independent. Also since $\alpha(x \otimes (f + f_0)) = B_f^N x \otimes h_f + B_{f_0}^N \otimes h_{f_0}$ is rank one, $B_f^N x$ and $B_{f_0}^N x$ are linearly dependent for each $x \in N$. Then B_f^N and $B_{f_0}^N$ are linearly dependent.

Let $B_f^N = \lambda_f B_{f_0}^N$. Then

$$\alpha(x \otimes f) = B_f^N x \otimes h_f = B_{f_0}^N x \otimes \lambda_f h_f.$$

Let $B_n = B_{f_0}^N$ and replace $\lambda_f h_f$ with h_f. Then $\alpha(x \otimes f) = B_N x \otimes h_f$, for every $x \in N$ and $f \in N_-^{\perp}$.

Remark : For each $x \in N, f \in N_-^{\perp}$, from Lemma 3.5 and Lemma 3.7, $\alpha(x \otimes f) = \varphi(x) \otimes C_N f = B_N x \otimes h_f$. By scalar multiplying, we can suppose that for each $N \in \mathcal{N}_*$, there exists $B_N \in B(N)$ and $C_N \in B(N_-^{\perp})$ such that $\alpha(x \otimes f) = B_N x \otimes C_N f$, for every $x \in N, f \in N_-^{\perp}$.

By the above arguments and the proof of Theorem 17.12 of[4], we have :

Lemma 3.8 — There exist linear mappings :

$$B : \bigcup \{N : N \in \mathcal{N}_*\} \to \bigcup \{N : N \in \mathcal{N}_*\}$$

$$C : \bigcup \{N_-^{\perp} : N \in \mathcal{N}_*\} \to \bigcup \{N_-^{\perp} : N \in \mathcal{N}_*\}$$

such that $\alpha(x \otimes f) = Bx \otimes Cf$, for any $x \otimes f \in \mathcal{T}(\mathcal{N})$ and $B|_N = B_N$, $C|_{N_-^{\perp}} = C_N$.

Remark : If $\mathcal{N}$ is a nest such that $0_+ \neq 0$ and $X_- \neq X$, then B and C are injective linear operators on X and X^*, respectively. By The Closed Graph Theorem, we can prove easily that B and C are bounded. In general case, B and C can extend uniquely to be bounded linear operators.

Lemma 3.9 — Let B, C be as in Lemma 3.9. Then B and C are bounded and injective on their domains. Furthermore, they can extend uniquely to be bounded linear operators on X and X^* with continuous inverses, respectively.

PROOF : If $Bx = 0$ for some $x \in N$ $(N \in \mathcal{N}_*)$, then take a nonzero vector $f \in N_-^{\perp}$, $\alpha(x \otimes f) = Bx \otimes Cf = 0$. Since α is bijective, $x \otimes f = 0$, which implies that $x = 0$. So B is injective on its domain.

Similarly, C is injective on its domain.

We claim that B and C can extend to be bounded linear operators on X and X^*, respectively.

In fact fix $N \in \mathcal{N}_*$.

By Lemma 3.8, $B|_N$ and $C|_{N_-}$ are continuous. Now we show that $C^{-1} : C_{N_-}^{\perp} \to N_-^{\perp}$ and $B^{1} : BN \to N$ are continuous. Take a unit vector $x_0 \in N$, for every nonzero $f \in N_-^{\perp}$, $\alpha(x_0 \otimes f) = Bx_0 \otimes Cf$. Then $x_0 \otimes f = \alpha^{-1}(Bx_0 \otimes Cf)$. Therefore $\|f\| = \|x_0 \otimes \phi\| \leq \|\alpha^{-1}\| \|Bx_0\| \|Cf\|$, which implies that C^{-1} is continuous on $CN_-^{\perp}$. Let $\kappa = \|C^{-1}|_{CN_-^{\perp}}\|$. Similarly, B^{-1} is continuous on BN.

For every $Q \in \mathcal{N}_*$, if $Q \subseteq N$, from $B|_N$ is continuous, then $B|_Q$ is continuous and $\|B|_Q\| \leq \|B|_N\|$. On the other hand, if $Q \supset N$, then for every $x \in N$ and taking $f \in Q_-^{\perp}$ such that $\|f\| = 1$, $\|Bx\| \|Cf\| = \|Bx \otimes Cf\| = \|\alpha(x \otimes f)\| \leq \|\alpha\| \|x\|$, and thus $\|Bx\| \leq \kappa \|\alpha\| \|x\|$.

Let $\sigma = \max(\kappa\|\alpha\|, \|B|_N\|)$. Then $\|Bx\| \leq \sigma \|x\|$, for every $x \in \bigcup\{Q \in \mathcal{N} : \varepsilon\, \mathcal{N}_*\}$, i.e. B is bounded on its domain. Also since $\bigcup \{Q \in \mathcal{N} : Q \in \mathcal{N}_*\}$ is dense in X, B can uniquely extend to be a bounded linear operator on X denoted by B again.

Similarly, by the continuity of α and B and noticing that $\bigcup \{N_-^{\perp} : N \in \mathcal{N}_*\}$ is dense in X^*, we can prove that C is bounded and can extend to be a bounded linear operator on X^* denoted by C again.

From the continuity of B, C and α^{-1}, B and C have continuous inverses.

Theorem 3.10 — *Every continuous in the strong operator topology local inner automorphism on a nest algebra acting a reflexive Banach space is an inner automorphism.*

PROOF : Let $\mathcal{N}, \alpha, B, C$ be as in Lemma 3.9. Then $\alpha(x \otimes f) = Bx \otimes Cf$, for each $x \otimes f \in \mathcal{T}(\mathcal{N})$. Let $C = D^*$, $D \in B(X)$. Then $\alpha(x \otimes f) = B(x \otimes f)D$, for each $x \otimes f \in \mathcal{T}(\mathcal{N})$. By the linearity of α, $\alpha(F) = BFD$, for each finite rank operator $F \in \mathcal{T}(\mathcal{N})$. Since α is continuous in the strong operator topology and the subalgebra of $\mathcal{T}(\mathcal{N})$ generated by all the rank one operators in $\mathcal{T}(\mathcal{N})$ is dense in $\mathcal{T}(\mathcal{N})$ in the strong operator topology[9], $\alpha(T) = BTD$ for each $T \in \mathcal{T}(\mathcal{N})$. In particular, $\alpha(I) = BD = I$, and thus $D = B^{-1}$. So α is an automorphism. Also since $B(N) \subseteq N$ and $C(N_-^{\perp}) \subseteq N_-^{\perp}$ for each $N \in \mathcal{N}_*$, $B, D \in \mathcal{T}(\mathcal{N})$. Hence α is inner.

For the remainder of this section, we consider the local automorphisms of the nest algebra $\mathcal{T}(\mathcal{N})$. Let α be a local automorphism of $\mathcal{T}(\mathcal{N})$, and suppose that α is continuous in the strong operator topology. Then α and α^{-1} preserve the maximal rank one operator subspaces of $\mathcal{T}(\mathcal{N})$.

Let $N_0 \in \mathcal{N}_*$, choose $x_0 \in N_0$ such that $N_0 = \wedge\{P \in \mathcal{N} \mid x_0 \in P\}$. Then $L_{x_0}^{N_0}$ is a maximal rank operator subspace of $\mathcal{T}(\mathcal{N})$, and from Proposition 3.1, $\alpha(L_{x_0}^{N_0}) = L_{\varphi(x_0)}^{M_0}$ or $\alpha(L_{x_0}^{N_0}) = R_{g(x_0)}^{M_0}$ for $M_0 \in \mathcal{N}_*$, $\varphi(x_0) \in M_0$, $g_{(x_0)} \in M_{0-}^{\perp}$. Obviously, $R_{g(x_0)}^{M_0}$ and $L_{\varphi(x_0)}^{M_0}$ are maximal.

Proposition 3.11 — *Let N_0, x_0 be as in above. Suppose $\alpha(L_{x_0}^{N_0}) = L_{\varphi(x_0)}^{M_0}$. Then*

(1) for each $N \in \mathcal{N}_*$ and each $x \in N$, there are $Q \in \mathcal{N}_*$ and $\varphi(x) \in Q$ such that $\alpha(L_x^N) \subseteq L_{\varphi(x)}^{Q}$;

(2) For each $x \in N_0$, there is $\varphi(x) \in M_0$ such that $\alpha(L_x^{N_0}) = L_{\varphi(x)}^{M_0}$;

(3) For each $N \in \mathcal{N}_*$, there is $M \in \mathcal{N}_*$ such that for any $x \in N$, there is $\varphi(x) \in M$ satisfying $\alpha(L_x^N) = L_{\varphi(x)}^{M}$.

(4) For each $N \in \mathcal{N}_*$, there is $M \in \mathcal{N}_*$ such that for any $f \in N_-^{\perp}$, there is $g(f) \in M_-^{\perp}$ satisfying $\alpha(R_f^N) = R_{g(f)}^{M}$

PROOF : (1) Firstly, we show that, for each $x \in N_0$, there are $Q \in \mathcal{N}_*$ and $\varphi(x) \in Q$ such that $\alpha(L_x^{N_0}) \subseteq L_{\varphi(x)}^{Q}$;

Otherwise, from Proposition 3.1, there is $x \in N_0$ such that $\alpha(L_x^{N_0}) \subseteq R_{g(x)}^{M}$ for some $M \in \mathcal{N}_*$. Obviously x and x_0 are linearly independent. Let

$$\alpha(x \otimes f) = \varphi(f) \otimes f(x),\ \alpha(x_0 \otimes f) = \varphi(x_0) \otimes g(f) \text{ for each } f \in N_{0-}^{\perp}.$$

Then there is f_0 such that $\varphi(f_0)$ and $\varphi(x_0)$ are linearly independent (for otherwise $dim \alpha L_x^{N_0}) = 1$), and then $g(x)$ and $f(f_0)$ are linearly dependent. Similarly, there is f_1 such that $g(f_1)$ and $g(x)$ are linearly independent, and then $\varphi(f_1)$ and $\varphi(x_0)$ are linearly dependent. It follows that there is some constant k such that $\alpha(x \otimes f_1) = k\,\alpha(x_0 \otimes f_0)$, which is impossible, for x and x_0 are linearly independent.

For each $N \in \mathcal{N}_*$, $N_0 \subset N$, $\alpha(L_{x_0}^N) \subseteq \alpha(L_{x_0}^{N_0}) = L_{\varphi(x_0)}^{M_0}$. Note that, in the proof of above paragraph, the condition "$\alpha(L_{x_0}^{N_0}) \subseteq L_{\varphi(x_0)}^{M_0}$" is only sufficient. Repeating the process, we can show that for each $x \in N$, $\alpha(L_x^N) \subseteq L_{\varphi(x)}^Q$ for some $Q \in \mathcal{N}_*$.

For each $N \in \mathcal{N}_*$, $N \subseteq N_0$ and let $x \in N$, from $\alpha(L_x^{N_0}) \subseteq \alpha(L_x^N)$ and the first paragraph, one can get that $\alpha(L_x^N) \subseteq L_{\varphi(x)}^M$ for some $M \in \mathcal{N}_*$.

(2) Claim that $\alpha(L_x^{N_0}) = L_{\varphi(x)}^{M_0}$, for each $x \in N_0$.

Without loss generality, let x and x_0 be linearly independent. From (1), $\alpha(L_x^{N_0}) \subseteq L_{\varphi(x)}^Q$. Hence $\varphi(x_0)$ and $\varphi(x)$ are linearly independent (or prove it!) Suppose that

$$Q = \wedge \{P \in \mathcal{N} \mid \varphi(x) \in P\}. \text{ Let}$$

$$\alpha(x \otimes f) = \varphi(x) \otimes h(f),\ \alpha(x_0 \otimes f) = \varphi(x_0) \otimes g(f) \text{ for each } f \in N_{0\_}{}^{\perp}.$$

Then $h(f)$ and $g(f)$ are linearly dependent for each $f \in N_{0\_}{}^{\perp}$. Since $\alpha(L_{x_0}^{N_0}) = L_{\varphi(x_0)}^{M_0}$, $\alpha(L_x^{N_0}) = \{\varphi(x) \otimes f \mid f \in M_{0\_}{}^{\perp}\}\ L_{\varphi(x)}^Q$. Hence $\varphi(x) \in M_0$, and thus $\alpha(L_x^{N_0}) = L_{\varphi(x)}^{M_0}$.

(3) Let $N \in \mathcal{N}_*$, choose $y_0 \in N$ such that $N = \wedge\{P \in \mathcal{N} \mid y_0 \in P\}$. Then $L_{y_0}^N$ is a maximal rank one operator subspace. From (1), $\alpha(L_{y_0}^N) = L_{\varphi(y_0)}^M$ for some $M \in \mathcal{N}_*$, and $L_{\varphi(y_0)}^M$ is maximal. By the similar argument to (1), (2) for $\alpha(L_{y_0}^N)$, we have $\alpha(L_y^N) = L_{\varphi(y)}^M$ for each $y \in N$.

(4) Choose $N \in \mathcal{N}_*$ and $f \in N_\_^{\perp}$ such that R_f^N is a maximal rank one operator subspace. Then $\alpha(R_f^N) = R_{g(f)}^M$ for some $M \in \mathcal{N}_*$. For otherwise, $\alpha(R_f^N) = L_{\varphi(f)}^Q$. From ($3^0$), $\alpha(L_x^N) = L_{\varphi(x)}^P$. So $\alpha(R_f^N \cap L_x^N) = L_{\varphi(f)}^Q \cap L_{\varphi(x)}^P$, which is impossible. By the similar qay to (1), (2), (3) with $\alpha(R_f^N)$, it is not difficult to get (4).

Remark : In the hypothesis of Proposition 3.11, one can show that for each $N \in \mathcal{N}_*$, there is $M \in \mathcal{N}_*$ such that for each $x \in N$ and for each $f \in N_\_^{\perp}$, there are $\varphi(x) \in M$ and $g(f) \in M_\_^{\perp}$ such that

$$\alpha(L_x^N) = L_{\varphi(x)}^M,\ \alpha(R_f^N) = R_{g(f)}^{M\ \perp}.$$

By repeating the process to prove the Lemma 3.4-Theorem 3.10 step by step, we can get an invertible operator $A \in B(X)$ such that $\alpha = adA$, i.e., α is an automorphism. If we suppose that $\mathcal{N}$ is a continuous nest of a separable Hilbert space H, we can deal with α by the Similarity Theorem. Define a bijective map θ from $\mathcal{N}$ onto itself as $\theta(0) = 0$, $\theta(N) = M$, $\theta(X) = X$ if $X = X_\_$, where M is as in above. Obviously θ preserves the order of the nest $\mathcal{N}$. Since $\mathcal{N}$ is continuous, by the similarity Theorem, there is an invertible operator $S \in B(H)$ such that $\theta(N) = SN$ for each $N \in \mathcal{N}$. So $\alpha(L_x^N) = L_{S(\varphi(x))}^{SN}$ for every $N \in \mathcal{N}_*$ and $x \in N$, where $\varphi(x) \in N$. Let $\alpha' = S^{-1}\alpha S$. Then

α' is a continuous strongly local automorphism, and satisfies the conditions of Lemma 3.3-Theorem 3.10. So we have the following theorem.

Theorem 3.12 — *Let α be a continuous strongly local automorphism of the nest algebra $\mathcal{T}(\mathcal{N})$ acting on a reflexive Banach space. Suppose $\alpha(L_{x_0}^{N_0}) = L_{\varphi(x_0)}^{M_0}$ for some a maximal rank one operator subspace $L_{x_0}^{N_0}$ of $\mathcal{T}(\mathcal{N})$. Then α is an automorphism.*

Let α be as in above. Suppose $\alpha(L_{x_0}^{N_0}) = R_{g(x_0)}^{M_0}$, where $x_0 = \wedge\{P \in \mathcal{N} | x_0 \in P\}$. By the similar way to Proposition 3.11, we can show that for each $N \in \mathcal{N}_*$, there is $M \in \mathcal{N}_*$ such that for each $x \in N$ and for each $f \in N_{0-}{}^{\perp}$, there are $\varphi(f) \in M$ and $g(x) \in M_-{}^{\perp}$ such that

$$\alpha(L_x^N) = R_{g(x)}^M,\ \alpha(R_f^N) = L_{\varphi(f)}^M.$$

Furthermore, if let $M = \theta(N)$, then $\theta(N_1) \subseteq \theta(N_2)$ while $N_2 \subseteq N_1$, for each $N_1, N_2 \in \mathcal{N}_*$.

Suppose $0_+ \neq 0$ in $\mathcal{N}$. Choose $x_0 \in 0_+$ and $f_0 \in X^*$ such that $f_0(x_0) = 1$. Then $L_{x_0}^{0+}$ is maximal rank one. Without loss generality, we can assume that $\alpha(x_0 \otimes f_0) = x_0 \otimes f_0$ (for otherwise, by the hypothesis, $\alpha(x_0 \otimes f_0) = A(x_0 \otimes f_0)A^{-1}$ for some $A \in B(X)$, where ad (A) is an automorphism of $\mathcal{T}(\mathcal{N})$, we can replace α with $\alpha' = A^{-1}\alpha A$.) We can claim that $\alpha(L_{x_0}^{0+}) = L_{x_0}^{0+}$.

For .otherwise, if $\alpha(L_{x_0}^{0+}) = L_{x_0}^{M}$, then $L_{x_0}^{M} = L_{x_0}^{0+}$ because of the maximality of $L_{x_0}^{0+}$; Suppose that $\alpha(L_{x_0}^{0+}) = R_{f_0}^{0+}$ by the choose of f_0. From above paragraphs, for each $n \in \mathcal{N}_*$, there is $M \in \mathcal{N}_*$ such that $M \subseteq 0_+$ and $\alpha(L_x^N) = R_{g(x)}^M$ for each $x \in N$, which is impossible because of the surjection of α.

Suppose that $X_- \neq X$ in $\mathcal{N}$. Choose $x_0 \notin X_-$ and $f_0 \in X_-{}^{X3}$ such that $f_0(x_0) = 1$. Then $L_{x_0}^{X}$ is maximal rank one. Without loss generality, we can assume that $\alpha(x_0 \otimes f_0) = x_0 \otimes f_0$. It follows that $\alpha(L_{x_0}^X) = L_{x_0}^X$. For otherwise, $\alpha(L_{x_0}^X) = R_{f_0}^X$ by the maximality of $L_{x_0}^X$. From above paragraphs, for each $N \in \mathcal{N}_*$ and $N \subset X$, $\alpha(L_x^N) = R_{g(x)}^X$ for each $x \in N$, which contradicts with the surjection of α.

By theorem 3.12, we have the following corollary :

Corollary 3.13 — Suppose that $0_+ \neq 0$ or $X_- \neq X$ in $\mathcal{N}$. Then every continuous strongly local automorphism of $\mathcal{T}(\mathcal{N})$ is an automorphism.

Question — Is the condition "$\alpha(L_{x_0}^{N_0}) = L_{\varphi(x_0)}^{M_0}$" in Theorem 3.12 in the general nest algebra in unnecessary ? In other words, does "$\alpha(L_{x_0}^{N_0}) = R_{g(x_0)}^{M_0}$" not occur in the hypothesis on the nest $\mathcal{N}$?

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