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SOME FIXED POINT THEOREMS OF INCREASING OPERATORS AND APPLICATIONS

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In this paper some new concepts such as increasing closed set, monotone closed set and cone-weakly compact set are introduced. The existence of the fixed points of a discontinuous increasing operator is discussed, and the form of the fixed points is also given. Finally, as an interesting application of the results in this paper, we discuss the existence of the coupled fxed points for mixed monotone operators.

Key Words : Increasing Operator; Fixed Point; Cone; Increasing Closed Set; Cone-Weakly Compact Set

1. INTRODUCTION

It is well known that the fixed points of increasing operators have important applications to investigate the solutions of nonlinear differential and integral eqs.[1-3]. Let E be a real Banach space, which is partially ordered by a cone P of E, $D = [u_0, v_0] := \{z \in E \mid u_0 \leq z \leq v_0\}$ be an ordered interval of E. Assume that $A : D \rightarrow D$ is an increasing operator satisfying

$$u_0 \leq Au_0, A_{v_0} \leq v_0.$$

On the fixed point theorems of increasing operators, as main theorem is :

Theorem A [1-3] — *Suppose that* $A : D \rightarrow D$ *is continuous and A(D) is a relatively compact subset in E. Then A has at least one fixed point.*

Theorem A requires that operator A is continuous and $A(D)$ is a relatively compact subset in E. But in many cases the conditions can not be satisfied. In 1986, Sun Jing Xian and Sun Yong proved the following conclusion without the continuity assumption by using Zorn's Lemma[4].

Assume that $A : D \rightarrow D$ is an increasing operator satisfying $u_0 \leq Au_0, A\, v_0 \leq v_0$ and $A(D)$ is a relatively compact set in E. Then A has at least one fixed point in D.

Zorn's Lemma plays an important role in proving the existence of fixed point, but it cannot give the form of a fixed point. In this paper, we discuss the existence of the fixed points of the discontinuous increasing operator, meanwhile, we also give the form of the fixed point by using a constructive method. This paper is organized as follows : in section 2, some new concepts such as increasing closed set, monotone closed set and cone-weakly compact set are introduced. Section 3 is devoted to discuss the existence of fixed points of increasing operators. Finally, as an interesting application of the results in this paper, we discuss the existence of the coupled fixed points for mixed monotone operators.

2. PRELIMINARIES

Let E be a real normed space, the topological dual space of E is denoted by E^*. A set $P \subset E$ is said to be a cone if

$$\lambda c \in P, \ \forall\, c \in P \text{ and } \lambda \geq 0,$$

and P is said to be a convex cone if in addition

$$P + P \subset P.$$

A cone P is said to be point if

$$P \cap (-P) = \{0\}.$$

Let P be a closed pointed convex cone, the partial order generated by P is defined as follows : for $x, y \in E, x \geq y$ if $x - y \in P$. Whenever $x \geq y$ and $x \neq y$, we write $x > y$. For a set $C \subset E$, clC denotes its topological closure and the cone generated by C, *cone* (C) is defined as

cone $(C) := \{\lambda c \mid \lambda \geq 0\}$.

Let $P^+ = \{f \in E^* \mid f(c) \geq 0, \ \forall\ c \in P\}$ denote the set of positive functionals, $P^{+i} = \{f \in E^* \mid f(c) > 0, \ \forall\ 0 \neq c \in P\}$ be the set of strictly positive functionals, and $P^{st} = \{f \in E^* \mid \exists\, m > 0$, such that $f(x) \geq m \| x \|, \ \forall\ x \in P\}$. P is said to admit strictly positive functional if P^{+i} is nonempty.

It is clear that

$$P^{st} \subset P^{+i} \subset P^+.$$

Recall that a base of cone P is a convex subset B of P such that

$$0 \notin clB \text{ and } P = cone\ (B).$$

Of course, C is pointed whenever C has a bse.

Remark 1[6] : When E is a topological vector space, a convex cone P has nonempty P^{+i} if and only if there exists an open convex set $U \subset E$ satisfying $0 \notin U$ and $P = cone\ (U)$. If E is a locally convex space, $P^{+i} \neq \phi$ if P has a base, the assumption of local convexity is unnecessary for the converse relation.

Lemma 1[7] — If E is a separable normed space and P is a pointed closed convex cone, then P^{+i} is nonempty.

Lemma 2[7] — If E is a normed space and P is a cone with a bounded base, then P^{st} is nonempty.

Below, we list some examples of cones with strictly positive functionals, (see also Ref. 5). Let $l^p, c_0, m, L_{[0,1]}, M_{[0,1]}$ be classical Banach spaces.

Example 1 — (*i*) Let $P = R^n_+$ be the cone of vectors with nonnegative components in R^n, and let

$$f(e) = \sum_{i=1}^{n} e_i \ , \ \forall e = \left\{ e_i \right\}_i^n \in R^n.$$

Then, f is strictly positive on P.

(*ii*) Let E be cone of the Banach space l^p, $1 \le p \le \infty$, and let P be the cone of vectors with nonnegative components in the corresponding space. Let f be the functionals defined by

$$f(e) = \sum_{i=1}^{\infty} e_i/i^2, \ \forall e = \left\{ e_i \right\}_i^\infty \in E.$$

Then, f is strictly positive on P.

(*iii*) Let P be the cone L^p_+ of a.e. nonnegative functions in $L^p_{[0,1]}$, $1 \le p < \infty$. Then, the functional

$$f(x) = \int_0^1 x(t)dt$$

is strictly positive on P.

(*iv*) Let E be the Banach space $C_{[0,1]}$. Then the functional

$$f(x) = \int_0^1 x(t)dt$$

is strictly positive on the cone $P^+_{[0,1]}$.

(*v*) Let E be one of the Banach spaces $M_{[0,1]}$, and let P be the cone of nonnegative increasing functions. Then the functional

$$f(x) = x(1)$$

is strictly positive on K.

Remark 2 : The previous examples show that the positive cone in many common Banach spaces admit strictly positive functionals. However, this is not always the case. For example, for the cone P of nonnegative bounded functions in the space $M_{[0,1]}$ of bounded functions defined on $[0, 1]$, P^{+i} is empty.

Let D be a nonempty subset of E, $A : D \to D$ is called an increasing operator if $x \le y$ $(x, y \in D)$ implies $Ax \le Ay$. $\bar{x} \in D$ is called a fixed point of operator A if $\bar{x} = A(\bar{x})$.

Definition 1 — (*i*) $D \subset E$ is said to be an increasing closed set if for any increasing sequence in D, whenever it is convergent, then its limit must belong to D.

(*ii*) $D \subset E$ is sad to be a monotone closed set if for any monotone sequence in D, whenever it is convergent, then its limit must belong to D.

Remark 3 : It is clear that a closed subset is an increasing closed subset, but the converse is not generally true.

Example 2 — Let $R = (-\infty, +\infty), M_1 = \{(1 - 1/n, 1 + 1/n) \mid n = 1, 2, \ldots\} \cup \{(2/3 - 1/n, 4/3 - 1/n) \mid n = 1, 2, \ldots\}$, set $P_1 = \{(x, y) \in R \times R \mid x \geq 0, y \geq 0\}$, the partial order generated by P_1 is defined as follows :

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2, y_1 \leq y_2.$$

Then M_1 is an increasing closed set, but it is not a closed set.

Definition 2 — (*i*) $D \subset E$ is said to be an increasing weakly closed set if for any increasing sequence in D, whenever it is weakly convergent, then its weakly limit must belong to D.

(*ii*) $D \subset E$ is said to be a monotone weakly closed set if for any monotone sequence in D, whenever it is weakly convergent, then its weakly limit must belong to D.

Definition 3 — $D \subset E$ is called a P-weakly compact set if for any $x \in D, (x + P) \cap D$ is a weakly compact set.

Remark 4 : It is not difficult to show that a weakly compact set is a P-weakly compact set, but in general, the converse is not true.

Example 3 — Let $R = (-\infty, +\infty), E = R \times R, P_1 = \{(x_1, x_2) \in E \mid x_1 \geq 0, x_2 \geq 0\}$,

$$M_2 = \{x \in E \mid x_1 + x_2 \leq 1\}.$$

It is clear that M_2 is not a compact set, but for any $x \in M_2, (x + P_1) \cap M_2$ is a bounded closed set, then it is a compact set. Hence M_2 is a P_1-weakly compact set.

Definition 4 — $D \subset E$ is clled a strongly P-weakly compact set if for any $x \in D$, both $(x + P) \cap D$ and $(x - P) \cap D$ are two weakly compact sets.

Remark 5 : Obviously, a weakly compact set is a strongly P-weakly compact set, but in general, the converse is not true.

Example 4 — Let $R = (-\infty, +\infty), E = R \times R, P_1 = \{(x_1, x_2) \in E \mid x_1 \geq 0, x_2 \geq 0\}$, $M_3 = \{x \in E \mid -1 \leq x_1 + x_2 \leq 1\}$.

It is clear that M_3 is not a compact set, but for any $x \in M_3\, (x + P_1) \cap M_3$ and $(x - P_1) \cap M_3$ are two bounded closed sets, then they are two compact sets. Hence, M_3 is a P_1-weakly compact set.

3. Main Results

Lemma 3 — Assue that $D \subset E$ and $A : D \to D$ is an increasing operator. Let $D \subset E$ be a nonempty increasing weakly closed set. Then $M = \{x \in D \mid Ax \geq x\}$ is an increasing weakly closed set.

Proof : If $m = \phi$, then the proof is completed. Assume that $M \neq \phi$, $\{x_n \mid n \in N\}$ is any increasing sequence of M, without loss of generality, let

$$x_n \xrightarrow{w} \bar{x}\,(n \to \infty). \quad \text{... (1)}$$

Since D is an increasing weakly closed set, then $\bar{x} \in D$. On the other hand, since $\{x_n \mid n \in N\}$ is an increasing sequence of M. Then for any $n \in N$, when $k \geq 0$, $x_n \leq x_{n+k}$, i.e., $x_{n+k} - x_n \in P$. Since P is a closed convex set, it is weakly closed. Hence we have $\bar{x} - x_n \in P$, i.e.,

$$x_n \leq \bar{x}, \ \forall\, n \in N. \quad \text{... (2)}$$

Since A is an increasing operator, $x_n \in M$ by (2). Then $x_n \leq Ax_n \leq A\,\bar{x}$, i.e.,

$$x_n \leq A\,\bar{x}, \ \forall\, n \in N. \quad \text{... (3)}$$

By (1), (3) and that P is a weakly closed set, we have $\bar{x} \leq A\,\bar{x}$, i.e., $\bar{x} \in M$. The proof is completed.

Now we prove the main result of this paper.

Theorem 1 — *Suppose that $D \subset E$ is a nonempty* P*-weakly compact set and P^{+i} is nonempty. Let $u_0 \in D$ and $A : D \to D$ be an increasing operator such that*

$$Au_0 \geq u_0. \quad \text{... (4)}$$

Then A has at least one fixed point in D.

PROOF : Let $M = \{x \in D \mid Ax \geq x\}$. It is clear that M is nonempty. Since D is a P-weakly compact set, by Lemma 3, M is an increasing weakly closed set.

For any $x_0 \in M$, if $Ax_0 = x_0$. then the proof is finished. Otherwise, $Ax_0 \geq x_0$. Setting $P^0 = \{x \in P \mid x \neq 0\}$, $D_1 = (x_0 + P^0) \cap M$, then $D_1 \neq \phi$ (Since $Ax_0 \in D_1$). Moreover, since D is P-weakly compact set. Then $D_1 = (x_0 + P^0) \cap M \subset (x_0 + P) \cap D$ is a bounded set. Taking $f \in P^{+i} \subset P^+$, let $d_1 = \sup f(D_1)$, then $d_1 < +\infty$. By the definition of supremum, there exists $x_1 \in D_1$ such that

$$f(x_1) > d_1 - 1/2.$$

If $Ax_1 = x_1$, the proof is finished. Otherwise, setting $D_2 := (x_1 + P^0) \cap M$, then $D_2 \neq \phi$. Let $d_2 = \sup f(D_2)$, then $d_2 < +\infty$. By the definition of supremum, there exists $x_2 \in D_2$ such that

$$f(x_2) > d_2 - 1/3.$$

Repeating the process in the same way, only two cases occur :

Case 1 — There exists an $x_n \in D_n$ such that $Ax_n = x_n$. The conclusion is true.

Case 2 — There exists a sequence $x_1, x_2, \ldots, x_n, \ldots$ with the following properties :

$$x_{n+1} \in D_{n+1} = (x_n + P^0) \cap M, n = 1, 2, \ldots$$

$$f(x_n) > d_n - \frac{1}{n+1}. \quad \text{... (5)}$$

$$d_n = \sup \{f(x) \mid x \in D_n\},\ n = 1, 2, \ldots . \qquad \ldots (6)$$

Since $x_{n+1} \in D_{n+1} = (x_n + P^0) \cap M$, it follows that

$$x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots, \qquad \ldots (7)$$

and $x_n \in (x_0 + P) \cap D$. Since $(x_0 + P) \cap D$ is a bounded set, hence $\{f(x_n)\}$ is a bounded sequence. Noting that $f \in P^{+i} \subset P^+$ and (7), we have $f(x_n) \leq f(x_{n+1}), n = 1, 2, \ldots$. Then $\{f(x_n) \mid n = 1, 2, \ldots\}$ is a bounded increasing sequence, thus there is a d such that

$$d = \lim_{n \to \infty} f(x_n) = \sup \{f(x_n) \mid n = 1, 2, \ldots\}.$$

On the other hand, $\{x_n\} \subset (x_0 + P) \cap D$. Since $(x_0 + P) \cap D$ is a weakly compact subset, then there exists a weakly convergent subsequence (Without loss of generality, we denote by $\{x_n\}$ the subsequence) and $\bar{z} \in D$ such that

$$x_n \xrightarrow{w} \bar{z}\ (n \to \infty).$$

Thus $d = \lim_{n \to \infty} f(x_n) = f(\bar{z})$. Since M is an increasing weakly closed set, by (7) we have $\bar{z} \in M$.

Now we will prove that $A\bar{z} = \bar{z}$. If not, then $A\bar{z} > \bar{z}$. Set $\bar{y} = A\bar{z}$, an argument similar to that in (2) of Lemma 3, we have

$$\bar{y} > \bar{z} \geq x_n,\ \forall n \in N.$$

Then $\bar{y} \in (x_n + P^0) \cap M = D_n$. By (5), (6) and $f \in P^+$, it implies that

$$f(x_n) > d_n - \frac{1}{n+1} \geq f(\bar{y}) - \frac{1}{n+1} \geq f(x_n) - \frac{1}{n+1}. \qquad \ldots (8)$$

Taking limit in (8) as $n \to \infty$, we have

$$f(\bar{z}) = f(\bar{y}). \qquad \ldots (9)$$

Noting that $\bar{y} > \bar{z}$ and $f \in P^{+i}$, then $f(\bar{y}) > f(\bar{z})$, which contradicts (9). Hence $\bar{z} = \bar{y}$, i.e., $A\bar{z} = \bar{z}$. The proof is completed.

Corollary 1 — Suppose that E is a separable normed space and $D \subset E$ is a nonempty P-weakly compact set. Let $u_0 \in D$ and $A : D \to D$ be an increasing operator such that (4) holds. Then A has at least one fixed point in D.

PROOF : By Lemma 1, we have that P^{+i} is nonempty. Then all conditions of theorem 1 are satisfied. According to Theorem 1, A has at least one fixed point in D.

***Theorem* 2** — *Assume that E is a Banach space, $D \subset E$ is a nonempty bounded increasing weakly closed set, and P is a cone with a bounded base. Let $u_0 \in D$ and $A : D \to D$ be an increasing operator such that (4) holds. Then A has at least one fixed point in D.*

PROOF : Since P is a cone with a bounded base, by Lemma 2 there exists a $f \in P^{st}$ such that

$$f(x) \geq m \| x \|, \ \forall x \in P, \quad \ldots (10)$$

where constant $m > 0$. Since $P^{st} \subset P^{+i}$, similar proof to that in Theorem 1, there are two cases to occur :

Case 1 — There exists an $x_n \in D_n$ such that $Ax_n = x_n$. The conclusion is true.

Case 2 — There exists a sequence $x_1, x_2, \ldots, x_n, \ldots$ with the following properties.

$$x_{n+1} \in D_{n+1} = (x_n + P^0) \cap M, n = 1, 2, \ldots .$$

$$f(x_n) > d_n - \frac{1}{n+1}.$$

$$d_n = \sup \{f(x) \mid x \in D_n\}, \ n = 1, 2, \ldots .$$

Since D is a bounded set, then $\{f(x_n) \mid n = 1, 2, \ldots\}$ is a bounded increasing sequence. Thus there is a d such that

$$d = \lim_{n \to \infty} f(x_n).$$

It suffices to show that there exists $\bar{x} \in D$ such that

$$x_n \xrightarrow{w} \bar{x} \ (n \to \infty).$$

In fact, let $n \geq m$,

$$x_n - x_m = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + .. + (x_{m-1} - x_m) \in P + P + \ldots + P \subset P,$$

by (10), we have

$$m \| x_n - x_m \| \leq f(x_n - x_m) = f(x_n) - f(x_m) \to 0 \ (n \to \infty).$$

Then $\{x_n\}$ is a cauchy sequence. Since E is a Banach space, there exists an $\bar{x} \in E$ such that $x_n \to \bar{x} \ (n \to \infty)$. Of course, $x_n \xrightarrow{w} \bar{x} \ (n \to \infty)$. Moreover, since D is an increasing weakly closed set, $\bar{x} \in D$. The proof is finished.

4. APPLICATIONS

Assume that D is a nonempty subset of a real normed space E, which is partial ordered by a cone P of E. Let $A : D \times D \to E$. A is said to be mixed monotone[3] if $A(x, y)$ is nondecreasing in x and nonincreasing in y, i.e., $x_1 \leq x_2$ $(x_1, x_2 \in D)$ implies $A(x_1, y) \leq A(x_2, y)$ for any $y \in D$ and $y_1 \leq y_2$ $(y_1, y_2 \in D)$ implies $A(x, y_1) \geq A(x, y_2)$ for any $x \in D$. Point $(\bar{x}, \bar{y}) \in D \times D$ is called a coupled fixed point of A if $A(\bar{x}, \bar{y}) = \bar{x}$ and $A(\bar{y}, \bar{x}) = \bar{y}$.

***Theorem* 3** — *Suppose that $D \subset E$ is a nonempty strongly P-weakly compact subset, and P^{+i} is nonempty. Let $u_0, v_0 \in D$ and $A : D \times D \to D$ be a mixed monotone operator such that*

$$A(u_0, v_0) \geq u_0, A(v_0, u_0) \leq v_0. \quad \ldots (11)$$

Then A has at least one coupled fixed point.

PROOF : Let $P_2 = \{(x, y) \in E \times E : x \geq 0, y \leq 0\}$. It is easy to see that P_2 is a cone in $E \times E$, therefore P_2 defines a partial ordering in $E \times E$ by

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2, y_1 \geq y_2. \quad \ldots (12)$$

The proof is divided into the following three steps.

Step 1 — Set $B(x, y) = (A(x, y), A(y, x))$, $\forall$ $(x, y) \in D \times D$, then $B : D \times D \to D \times D$ is an increasing operator such that

$$B(u_0, v_0) \geq (u_0, v_0).$$

In fact, let $(x_1, y_1) \leq (x_2, y_2)$. Since A is a mixed monotone operator, by (12) it implies that

$$B(x_1, y_1) \leq B(x_2, y_1) \leq B(x_2, y_2).$$

Moreover, $$B(u_0, v_0) = (A(u_0, v_0), A(v_0, u_0)) \geq (u_0, v_0).$$

Step 2 — $P_2^{+i} \neq \phi$. Indeed, since $P^{+i} \neq \phi$, taking $f \in P^{+i}$, let $f = (f, -f)$, then $g \in P_2^{+i}$.

Step 3 — $D \times D$ is a P_2-weakly compact set.

It suffices to show that for any $(x, y) \in D \times D$, $((x, y) + P_2) \cap (D \times D)$ is a weakly compact set. Indeed, for any sequence $(x_n, y_n) \in ((x, y) + P_2) \cap (D \times D)$ there exists $c_n \in P$, $d_n \in -P$ such that $x_n = x + c_n$, $y_n = y + d_n$. Hence $x_n \in (x + P) \cap D$, $y_n \in (y - P) \cap D$. Since D is a strongly P-weakly compact set, there exists two weakly convergent subsequence. (Without loss of generality, we denote them respectively by $\{x_n\}$ and $\{y_n\}$) and $\bar{x} \in (x + P) \cap D$, $\bar{y} \in (x - P) \cap D$ such that $x_n \xrightarrow{w} \bar{x}$, $y_n \xrightarrow{w} \bar{y}$ $(n \to \infty)$. hence $(x_n, y_n) \xrightarrow{w} (\bar{x}, \bar{y}) \in ((x, y) + P_2) \cap (D \times D)$. Thus all conditions of theorem 1 are satisfied. By Theorem 1, there exists $(\bar{u}, \bar{v}) \in D \times D$ such that $B(\bar{u}, \bar{v}) = (\bar{u}, \bar{v})$, i.e., $A(\bar{u}, \bar{v}) = \bar{u}$, $A(\bar{v}, \bar{u}) = \bar{v}$. The proof is completed.

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