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ON GENERALIZED STATISTICALLY CONVERGENT SEQUENCES

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This article introduces $\bar{c}(p)$ and $\bar{c}_0(p)$ for $p = (p_k)$ a sequence of non-negative real numbers. This generalizes the spaces of statistically convergent and statistically null sequences. Some properties of these spaces are studied.

Key Words : Density; Separable; Statistically Convergent; Statistically Null; Paranorm

1. INTRODUCTION

In order to extend the notion of convergence of sequences, statistical convergence was introduced by Fast[2], Schoenberg[14] and Zygmund[19] independently. Later on it was studied and linked with summability by Fridy[3, 4], Connor[1], Šalát[13], Kolk[5], Rath and Tripathy[11 & 12], Tripathy[16, 17 & 18] and many others. The idea depends on the density of subsets of the set N of natural numbers. A subset A of N is said to have *density* $\delta(A)$ if

$$\delta(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k) \text{ exists.}$$

where X_A is the characteristic function of A.

Clearly finite subsets of N have zero natural density and $\delta(A^c) = \delta(N - A) = 1 - \delta(A)$. For (x_k) and (y_k) two sequences, we say that $x_k = y_k$ for *almost all* k (in short a.a.k) if $\delta(\{k \in N : x_k \neq y_k\}) = 0$.

A sequence (x_k) is said to be *statistically convergent* to L if for any $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$ and we write $x_k \xrightarrow{stat} L$.

A sequence (x_k) is said to be a *statistically Cauchy sequence* provided that for every $\varepsilon > 0$ there exists a number n_0 such that $|x_k - x_{n_0}| < \varepsilon$ a.a.k.

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A subsequence (x_{k_n}) of (x_n) is said to be a *thin* subsequence of (x_n) if $\delta(\{k_n : n \in N\}) = 0$. It is said to be *non-thin* subsequence if either $\{k_n : n \in N\}$ has non-zero density or fails to have density.

2. Definitions and Background

Throughout the article $s, l_\infty, c, c_0, l_1, l_p, \bar{c}, \bar{c}_0, m, m_0$ will denote the spaces of *all, bounded, convergent, null, absolutely summable, p-absolutely summable, statistically convergent, statistically null, bounded statistically convergent* and *bounded statistically null* sequences respectively. Further $p = (p_k)$ will denote a sequence of strictly positive numbers (not necessarily bounded in general).

The concept of generalized sequences was studied by Nakano[10] and Simons[15] at the initial stage. Later on it was studied by Maddox[8 & 9], Lascarides[6 & 7] and many others. Generalizing l_∞, c, c_0, l_p people have studied $l_\infty(p), c(p), c_0(p), l(p)$ etc. and have characterized some matrix classes.

Let $p = (p_k)$ be a real sequence and $H = \sup_k p_k < \infty$.

Then we have the following well known inequality

$$|a_k + b_k|^{p_k} \le C(|a_k|^{p_k} + |b_k|^{p_k}), \qquad \text{... (2.1)}$$

where $C = \max(1, 2^{H-1})$.

On generalizing the sequence spaces $\bar{c}$ and $\bar{c}_0$ we define the following spaces

$$\bar{c}(p) = \{(x_k) \in S : |x_k - L|^{p_k} \overset{stat}{\rightarrow} 0, \text{ as } k \to \infty \text{ for some } L\}$$

$$\bar{c}_0(p) = \{(x_k) \in S : |x_k|^{p_k} \overset{stat}{\rightarrow} 0, \text{ as } k \to \infty\}.$$

With the help of inequality (2.1), one can easily see that $m(p) = \bar{c}(p) \cap l_\infty(p)$ is an additive group, the addition being co-ordinatewise addition. From the existing results, we find that $l_\infty(p), c(p), c_0(p), l(p)$ are linear spaces if and only if $(p_k)\ l_\infty$. It can be easily verified that $m(p)$ and $m_0(p)$ are linear spaces if and only if $(p_k) \in l_\infty$. The spaces $l_\infty(p), c(p), c_0(p)$ are paranormed by

$$g(x) = \sup_k |x_k|^{\frac{p_k}{M}}$$

and $l(p)$ by

$$f(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M},$$

where $M = \max(1, H)$. It is clear that the spaces $m(p)$ and $m_0(p) = \bar{c}_0(p) \cap l_\infty(p)$ are also paranormed by g.

The following results will be used for establishing the results of this article.

Lemma 1 (Fridy[3], Theorem 1) — The following statements are equivalent.

(*i*) (x_k) is a statistically convergent sequence;

(ii) (x_k) is a statistically Cauchy sequence;

(iii) (x_k) is a sequence for which there is a convergent sequence (y_k) such that $x_k = y_k$ a.a.k.

Lemma 2 (Šalát[13], lemma 1.1) — A sequence (x_k) statistically converges to L if and only if there exists such a set $K = \{k_1 < k_2 < \ldots < k_n < \ldots\} \subset N$ that $\delta(K) = 1$ and $\lim_{n \to \infty} x_{k_n} = L$.

Lemma 3 (Connor[1], Theorem 2.3) — If $x \in s$ is statistically convergent to L, then there is a convergent sequence y and *a* statistically null sequence z such that y is convergent to L, $x = y + z$ and $\delta(\{k \in N : z_k \neq 0\}) = 0$. Moreover, if x is bounded then z is bounded and $\| z \|_\infty \leq \| x \|_\infty + | L |$.

3. Main Results

In this section we prove some results relating to $m(p)$ and $m_0(p)$.

The following results is the decomposition theorem for $m(p)$. For $p_k = 1$, for all $k \in N$ we have the decomposition theorem earlier proved by Fridy[3], Šalàt[13] and Connor[1] for m and m_0.

Theorem 1 — *Let* $0 < \inf p_k \leq p_k \leq \sup p_k < \infty$, *then the following are equivalent.*

(a) $(x_k) \in m(p)$

(b) $(x_k - L) \in m_0(p)$ *for some* L

(c) *there exists* $(y_k) \in c(p)$ *such that* $x_k = y_k$ *for a.a.k*

(d) *there exists a subset* $K = \{k_1, k_2, \ldots\ldots\}$ *of* N *such that* $\delta(K) = 1$ *and* $\lim_{n \to \infty} | x_{k_n} - L |^{p_{k_n}} = 0$

(e) *there exists sequences* (y_k) *and* (z_k) *such that* $x_k = y_k + z_k$ *for all* $k \in N$ *and* $(y_k) \in c(p)$, $(z_k) \in m_0(p)$.

PROOF : The equivalence of (*a*) and (*b*) is clear from the definitions.

Let $(x_k) \in m(p)$. Then there exists L such that $| x_k - L |^{p_k} \xrightarrow{stat} 0$. Let $z_k = | x_k - L |^{p_k}$, then $(z_k) \in m_0(p)$. Now the equivalence of (a) and (c) follows from lemma 1, that of (a) and (d) from Lemma 2 and that of (a) and (e) from lemma 3.

Remark : We have the condition $0 < \inf p_k \leq p_k \leq \sup p_k < \infty$ on (p_k) in the above theorem. The condition $\sup p_k < \infty$ is necessary for the linearity of the space $c(p)$ (see Maddox[9]). We have established the equivalence of the relations involving $c(p)$. Hence, $\sup p_k < \infty$ is necessary. Further

Simons[15] proved that $l_\infty(p)$ is paranormed by $g(x) = \sup_k |x_k|^{\frac{p_k}{M}}$ only if $\inf p_k > 0$. So from the definition of $m(p)$, $\inf p_k > 0$ is necessary.

***Theorem* 2** — *$m(p)$ is a closed subspace of $l_\infty(p)$.*

PROOF : Let $(x^{(n)})$ be a Cauchy sequence in $m(p)$ such that $x^{(n)} \to x$ in $l_\infty(p)$.

We show that $x \in m(p)$. Since $x^{(n)} \in m(p)$, so there exists a_n such that

$$|x_k^{(n)} - a_n|^{p_k} \overset{stat}{\to} 0, \text{ as } k \to \infty.$$

To show that

(*i*) (a_n) converges to a

and (*ii*) $|x_k - a|^{p_k} \overset{stat}{\to} 0.$

(i) Since $(x^{(n)})$ is a convergent sequence of elements, so for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$\sup_k |x_k^{(n)} - x_k^{(m)}|^{\frac{p_k}{M}} < \frac{\varepsilon}{3} \text{ for all } m, n \geq n_0.$$

Again given $\varepsilon > 0$, we have

$$\delta(A_m) = \delta\left(\left\{k \in N : |x_k^{(m)} - a_m|^{p_k} < \left(\frac{\varepsilon}{3}\right)^M\right\}\right) = 1$$

and

$$\delta(A_n) = \delta\left(\left\{k \in N : |x_k^{(n)} - a_n|^{p_k} < \left(\frac{\varepsilon}{3}\right)^M\right\}\right) = 1.$$

Let $A = A_m \cap A_n$, then $\delta(A) = 1$. We choose $k \in A$. Then for each $m, n \geq n_0$, we have

$$|a_m - a_n| \leq |a_m - x_k^{(m)}| + |x_k^{(m)} - x_k^{(n)}| + |x_k^{(n)} - a_n|.$$

Thus (a_n) is a Cauchy sequence of scalars, so there exists a scalar a such that $\lim_{k \to \infty} a_k = a$.

(ii) Let $\eta > 0$, we show that

$$\delta(F) = \delta(\{k \in N : |x_k - a|^{p_k} < \eta\}) = 1.$$

Since $x^{(n)} \to x$, there exists $q \in N$ such that

$$\| (x_k^{(q)}) - (x_k) \| < \left(\frac{\eta}{3} \right)^{1/M} \qquad \ldots (3.1)$$

The number q can be chosen in such a way that together with (3.1) we have

$$| a_q - a |^{p_k} < \frac{\eta}{3} \qquad \ldots (3.2)$$

Again since $| x_k^q - a_q |^{p_k} \overset{stat}{\rightarrow} 0$, we have a subset B of N such that $\delta(B) = 1$,

where $$B = \left\{ k \in N : | x_k^{(q)} - a_q |^{p_k} < \frac{\eta}{3} \right\}.$$

Therefore, for each $k \in B$ we have

$$| x_k - a |^{p_k} \leq C^2 | x_k - x_k^{(q)} |^{p_k} + C^2 | x_k^{(q)} - a_q |^{p_k} + C | a_q - a |^{p_k}$$

$$< C^2 \left(\frac{\eta}{3} \right) + C^2 \left(\frac{\eta}{3} \right) + C \left(\frac{\eta}{3} \right) = \eta' \text{ (say)}.$$

This completes the proof.

***Theorem* 3** — *For two sequences* (p_k) *and* (q_k) *we have* $m_0(p) \supseteq m_0(q)$ *if and only if* $\liminf_{k \in k} (p_k/q_k) > 0$, *where* $K \subseteq N$ *such that* $\delta(K) = 1$.

PROOF : Let $$\liminf_{k \in K} \left(\frac{p_k}{q_k} \right) > 0 \qquad \ldots (3.3)$$

and $(x_k) \in m_0(q)$. Then there exists $\alpha > 0$ such that $p_k > \alpha q_k$ for all sufficiently large $k \in K$. Also $| x_k |^{q_k} \overset{stat}{\rightarrow} 0$ since $(x_k) \in m_0(q)$. Hence for $\varepsilon > 0$, we have $\delta(L) = 1$, where $L = \{ k \in N : | x_k |^{q_k} < \varepsilon \}$. Let $J = K \cap L$. Then $\delta(J) = 1$. Also for all sufficiently large $k \in J$, we have

$$| x_k |^{p_k} \leq (| x_k |^{q_k})^{\alpha}$$

$$\Rightarrow (x_k) \in m_0(p).$$

Conversely, let $m_0(q) \subseteq m_0(p)$ but there exists no $K \subset N$ with $\delta(K) = 1$ such that (3.3) holds. Then we get $k_1 < k_2 < k_3 < \ldots\ldots.$ with $\delta(\{k_i : i \in N\}) \neq 0$ such that $i p_{k_i} < q_{k_i}$. Let us define the sequence (x_k) as follows :-

$$x_k = \begin{cases} (1/i)^{\frac{1}{q k_i}}, & \text{if } k = k_i \\ 0, & \text{otherwise.} \end{cases}$$

Then $(x_k) \in m_0(q)$. But we have

$$(x_{k_i})^{p_{k_i}} > \exp\left(-\frac{\log i}{i} \right)$$

This contradicts the fact that $(x_k) \in m_0(p)$. This completes the proof.

From the above result we have the following.

Corollary — For (p_k) and (q_k) two non-negative sequence of reals, we have $m_0(p) = m_0(q)$ if and only if there exists a subset K of N with $\delta(K) = 1$ such that

$$\liminf_{k \in K} \left(\frac{p_k}{q_k}\right) > 0 \text{ and } \liminf_{k \in K} \left(\frac{q_k}{p_k}\right) > 0.$$

We have the following result which is an easy consequence of $c(p)$ and so we omit the proof.

***Theorem* 4** — *Let* $h = \inf_k p_k$, *then the following are equivalent :*

(i) $H < \infty$ and $h > 0$

(ii) $m(p) = m$.

4. Properties of $\bar{c}$

The proof of the following lemma is a routine work.

Lemma — Let $K = \{m_1, m_2, ...\}$ be an infinite subset of N such that $\delta(K) = 0$.

Let $L = \{(x_k) : x_k = 0 \text{ or } 1 \text{ for } k = m_i, i \subset N \text{ and } x_k = 0 \text{ otherwise}\}$. Then L is uncountable.

The proof of the following result is obvious in view of the above lemma.

***Theorem* 5** — *m is not separable.*

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