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PERIODIC BOUNDARY VALUE PROBLEMS AND MONOTONE ITERATIVE METHODS FOR FIRST ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH DELAY

FENGQIN ZHANG, ZHIEN MA

Department of Applied Mathematics, Xi'an Jiaotong University, Xi'an 710049, *People's Republic of China*

AND

JURANG YAN

Department of Mathematics, Shanxi University, Tai Yuan 030006, *People's Republic of China*

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This paper investigates periodic boundary value problems for first order impulsive differential equations with delay. By establishing comparison results, criteria on the existence of maximal and minimal solutions are obtained. The results of[10 & 12] are extended.

Key Words : Impulsive Differential Equation with Delay; Upper and Lower Solutions; Monotone Iterative Methods

1. INTRODUCTION

Recently, the investigation of impulsive differential equations has attracted much attention since many evolution processes resulting from applications are subject to short term perturbations in the form of impulses[1-3]. Furthermore, the theory of such equations is much richer than the corresponding theory of differential equations without impulses. There exist several papers about initial and boundary value problems with impulsive effects at fixed points[4-11], by means of the different techniques: limit arguments, topological degree, fixed point theorems, set-valued maps or upper and lower solutions and monotone iterative technique. In this paper, using the monotone iterative technique and the method of upper and lower solutions we extend the results of[10 & 12] to establish new results of the existence of the PBVP periodic boundary value problem for first order impulsive differential equations with delay.

2. MAIN RESULTS

Consider the PBVP for impulsive differential equation with delay

$$x'(t) = f(t, x(t), x_t),\ t \neq t_k,\ t \in [0, 2\pi]\,, \qquad \text{... (2.1)}$$

$$\Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \ldots, m \qquad \ldots (2.2)$$

and $$x(0) = x(2\pi), \qquad \ldots (2.3)$$

where $f \in C[[0, 2\pi] \times R \times B, R]$, $B = \{\varphi : [-\tau, 0] \to R : \varphi(t)$ is continuous everywhere except for a finite number of points t at which $\varphi(t^-)$ and $\varphi(t^+)$ exist and $\varphi(t) = \varphi(t^-)\}$, $I_k \in C[R, R]$, $\Delta x(t_k)$ represents the jump of $x(t)$ at $t = t_k$, i.e., $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for all $k = 1, 2, \ldots, m$, $0 < t_1 < t_2 \ldots < t_m < 2\pi$, $\tau > 0$, for every $t \in [0, 2\pi]$, $x_t \in B$ is defined by $x_t(s) = x(t + s)\ (-\tau \le s \le 0)$.

Suppose $J \subset R$ be an interval. Let

$PC[J, R] = \{x : J \to R, x(t)$ is continuous for $t \in J, t \ne t_k$ and $x(t_k^+), x(t_k^-)$ exist and $x(t_k) = x(t_k^-)$ for $k = 1, 2, \ldots, m\}$;

$PC'[J, R] = \{x : \in PC[J, R], x(t)$ is continuously differential for $t \in J, t \ne t_k$ and $x'(t_k^-), x'(t_k^+)$ exist and $x'(t_k) = x'(t_k^-)$ for $k = 1, 2, \ldots, m\}$;

$$E = PC[[-\tau, 2\pi], R] \cap PC'[[0, 2\pi], R];$$

$$E_0 = \{x \in PC[[-\tau, 2\pi], R] : x(\theta) \equiv x(0), \theta \in [-\tau, 0]\};$$

and $$E_1 = \{x \in PC[[-\tau, 2\pi], R] : x(\theta) \text{ is continuously differential for } -\tau \le \theta \le 0\}.$$

A function $\alpha(t) \in E$ is called a lower solution relative to (2.1)-(2.3) provided

$$\alpha'(t) \le f(t, \alpha(t), \alpha_t),\ t \ne t_k,\ t \in [0, 2\pi]$$

$$\Delta \alpha(t_k) \le I_k(\alpha(t_k)),\ k = 1, 2, \ldots, m$$

$$\alpha(0) \le \alpha(2\pi).$$

The inequality above is reversed for a upper solution.

Let $\alpha, \beta \in E$ be lower and upper solutions relative to (2.1)-(2.3) with

$$\alpha(t) \le \beta(t) \qquad \ldots (2.4)$$

In order to develop the monotone method for the PBVP (2.1)-(2.3), we require that f, I_k satisfies hypotheses (A_0) and (A_1) :

(A_0) There exist constant $M > 0$, $N \ge 0$, $L_k \in [0, 1)$ $(k = 1, 2, \ldots, m)$ such that

$$f(t, v, \phi) - f(t, u, \varphi) \ge -M(v - u) - N\int_{t-\tau}^{t} (\phi(\varphi) - \varphi(\theta)\, d\theta$$

and $$I_k(v) - I_k(u) \ge -L_k(v - u)$$

whenever $t \in [0, 2\pi]$, $\alpha(t) \le u \le v \le \beta(t)$, $\alpha_t(\theta) \le \varphi(\theta) \le \phi(\theta) \le \beta_t(\theta)$ for all $\theta \in [-\tau, 0]$.

(A_1) Constants M, N, L_k $(k = 1, 2, ..., m)$ satisfy the inequalities

$$M^{-1} N (e^{2+\pi M} + 1)\, \delta \leq \frac{\left\{ \sum_{k=1}^{m} (1 - L_k) \right\}^2}{1 + \sum_{n=1}^{m} \prod_{k=n}^{m} (1 - L_k)} \quad \text{... (2.5)}$$

and

$$N(2\pi + \tau) \left[\sum_{i=0}^{m-1} \prod_{t_i < t_k < 2\pi} (1 - L_k) \exp(-M(2\pi - t_{i+1})) \right]$$

$$< M \left(1 - \prod_{k=1}^{m} (1 - L_k) \exp(-2\pi M) \right) \quad \text{... (2.6)}$$

where $t_0 = 0, t_{m+1} = 2\pi$ and $\delta = \max \{t_k - t_{k-1} : k = 1, 2, ..., m + 1\}$

Now, we can state our main results.

***Theorem* 2.1** — *Suppose that $\alpha, \beta \in E \cap E_1$ are lower and upper solutions respectively relative to (2.1)-(2.3) satisfying (2.4). Let (A_0) hold and $N = 0$. Then there exist monotone sequences of functions $\{a_n(t)\}$ and $\{b_n(t)\}$ in $E \cap E_1$, with $a_0(t) = \alpha$, $b_0(t) = \beta(t)$ such that*

$$\lim_{n \to \infty} a_n(t) = \rho(t), \quad \lim_{n \to \infty} b_n(t) = r(t)$$

piecewise uniformly and monotonically on $[-\tau, 2\pi]$, and that $\rho(t)$, $r(t)$ minimal and maximal solutions of the PBVP (2.1)-(2.3).

Remark : Theorem 2.1 includes Theorem 2.2 in[12] as a special case when $I_k = 0$ $(k = 1, 2, ..., m)$, $x(t) \equiv x(0), t \in [-\tau, 0]$. Theorem 2.1 also includes theorem 1 in[10] as a special case when $x(t) \equiv x(0), t \in [-\tau, 0]$.

***Theorem* 2.2** — *Suppose that $\alpha, \beta \in E \cap E_0$ are lower and upper solutions respectively relative to (2.1)-(2.3) and*

$$x(t) \equiv x(0), \; t \in [-\tau, 0]$$

satisfying (2.4). Let (A_0) and (A_1) hold. Then there exist monotone sequences $\{a(t)\}$, $\{b_n(t)\}$ with $a_0(t) = \alpha(t)$, $b_0(t) = \beta(t)$ such that

$$\lim_{n \to \infty} a_n(t) = \rho(t), \quad \lim_{n \to \infty} b_n(t) = r(t)$$

piecewise uniformly on $[-\tau, 2\pi]$, and $\rho(t)$, $r(t)$ are the minimal and the maximal solutions of the PBVP (2.1)-(2.3), (2.7).

3. PROOF OF THEOREM 2.1

We shall give a comparison lemma.

Lemma 3.1[2] — Assume that $p(t) \in E$ satisfies

$$p'(t) \leq -Mp(t),\ t \neq t_k,\ t \in [0, 2\pi]$$

$$\Delta p(t_k) \leq -L_k\, p(t_k),\ k = 1, 2, \ldots, m$$

and $$p(0) \leq p(2\pi),$$

where constants $M > 0$, $L_k \in [0, 1)$ $(k = 1, 2, \ldots, m)$. Then $p(t) \leq 0$ for $t \in [0, 2\pi]$.

Proof of theorem 2.1. We consider the PBVP

$$u'(t) + Mu(t) = M\,\alpha(t) + f(t, \alpha(t), \alpha_t),\ t \neq t_k\quad t \in [0, 2\pi], \qquad \ldots (3.1)$$

$$\Delta u(t_k) = -L_k\, u(t_k) + I_k(\alpha(t_k)) + L_k\, \alpha(t_k),\ k = 1, 2, \ldots, m \qquad \ldots (3.2)$$

and $$u(0) = u(2\pi). \qquad \ldots (3.3)$$

For any $t \in [0, 2\pi]$, we have

$$u(t) = u(0) \prod_{0 < t_k < t} (1 - L_k) \exp(-Mt) + \int_0^t \prod_{S < t_k < t} (1 - L_k) \exp[-M(t - s)]\, \sigma(s) ds \qquad \ldots (3.4)$$

$$+ \sum_{0 < t_k < t}\ \prod_{t_k < t_i < t} (1 - L_i) \exp[-M(t - t_k)]\, [I_k(\alpha(t_k)) + L_k(\alpha(t_k)],$$

where $\sigma(s) = M\alpha(s) + f(s, \alpha(s), \alpha_s)$. Let $t = 2\pi$ in (2.4) we get

$$u(2\pi) = u(0) \prod_{0 < t_k < 2\pi} (1 - L_k) \exp(-2\pi M)$$

$$+ \int_0^{2\pi} \prod_{S < t_k < 2\pi} (1 - L_k) \exp[-M(2\pi - s)]\, \sigma(s) ds$$

$$+ \sum_{0 < t_k < 2\pi}\ \prod_{t_k < t_i < 2\pi} (1 - L_i) \exp[-M(2\pi - t_k)]\, [I_k(\alpha(t_k)) + L_k(\alpha(t_k)]$$

by (3.3) and $M > 0$, $0 \leq L_k < 1$ we obtain

$$u(2\pi) = u(0) = \left[1 - \prod_{k=1}^{m} (1 - L_k) \exp(-2\pi M) \right]^{-1}$$

$$\times\left[\int_{0}^{2\pi} \prod_{S<t_k<2\pi} (1-L_k)\exp[-M(2\pi-s)]\,\sigma(s)\,ds \right.$$

$$\left. + \sum_{0<t_k<2\pi} \prod_{t_k<t_i<2\pi} (1-L_i)\exp[-M(2\pi-t_k)]\,[I_k(\alpha(t_k))+L_k(\alpha(t_k)]\right].$$

Therefore, there exists a solution $u(t) \in E$ for the PBVP (3.1)-(3.3).

The uniqueness of solutions of the PBVP (3.1)-(3.3) follows from lemma 3.1. In fact, if $u, v \in E$ are two distinct solutions of the PBVP (3.1)-(3.3), setting

$$p(t) = u(t) - v(t),\ t \in [-\tau, 2\pi],$$

we see that
$$p'(t) = u'(t) - v'(t) = -Mp(t),\ t \in [0, 2\pi],\ t \neq t_k$$

$$\Delta p(t_k) = \Delta u(t_k) - \Delta v(t_k) = -L_k p(t_k),\ k = 1, 2, \ldots, m$$

$$p(0) = p(2\pi).$$

By means of lemma 3.1, $p(t) \leq 0$ on $[-\tau, 2\pi]$, which implies $u(t) \leq v(t)$ on $[-\tau, 2\pi]$. On the other hand, setting $p(t) = v(t) - u(t)$ and following the same argument, we can show that $v(t) \leq u(t)$ on $[-\tau, 2\pi]$. Thus $u(t) \equiv v(t)$ on $[-\tau, 2\pi]$. Therefore tha PBVP (3.1)-(3.3) has a unique solution called $a_1(t)$. Moreover, we have that

$$\alpha(t) \leq a_1(t) \text{ for all } t \in [-\tau, 2\pi] \qquad \ldots (3.5)$$

In fact, set $p = \alpha(t) - a_1(t)$, then we have

$$p'(t) = \alpha'(t) - a_1^u 2\text{–}'(t) \leq -Mp(t),\ t \neq t_k,\ t \in [0, 2\pi]$$

$$\Delta p(t_k) = \Delta\alpha(t_k) - \Delta a_I(t_k) \leq -L_k p(t_k),\ k = 1, 2, \ldots, m$$

$$p(0) \leq p(2\pi).$$

Hence, by lemma 3.1, we deduce that $p(t) \leq 0$ on $[-\tau, 2\pi]$, and this shows (3.5).

Similarly, one can find a unique solution b_1 of the PBVP

$$u'(t) + Mu(t) = \sigma_1(t),\ t \neq t_k,\ t \in [0, 2\pi],$$

$$\Delta u(t_k) = -L_k u(t_k) + I_k(\alpha(t_k)) + L_k \alpha(t_k),\ k = 1, 2, \ldots, m$$

$$u(0) = u(2\pi)$$

where $\sigma_1(t) = M\beta(t) + f(t, \beta(t), \beta_t)$. Moreover, by using assumptions of Theorem 2.1 and Lemma 3.1, we deduce that

$$\alpha(t) \leq a_1(t) \leq b_1(t) \leq \beta(t) \text{ on } [-\tau, 2\pi].$$

Now we consider sequences of PBVPS

$$a_n'(t) + Ma_n(t) = Ma_{n-1}(t) + f(t, a_{n-1}(t), a_t^{n-1}),\ t \neq t_k,\ t \in [0, 2\pi]$$

$$\Delta a_n(t_k) = -L_k a_n(t_k) + I_k(a_{n-1}(t_k)) + L_k a_{n-1}(t_k),\ k = 1, 2, \dots m \qquad \dots (3.6)$$

$$a_n(0) = a_n(2\pi)$$

and

$$b_n'(t) + Mb_n(t) = Mb_{n-1}(t) + f(t, b_{n-1}(t), b_t^{n-1}),\ t \neq t_k,\ t \in [0, 2\pi]$$

$$\Delta b_n(t_k) = -L_k b_n(t_k) + I_k(b_{n-1}(t_k)) + L_k b_{n-1}(t_k),\ k = 1, 2, \dots m \qquad \dots (3.7)$$

$$b_n(0) = b_n(2\pi).$$

Obviously, (3.6) and (3.7) have unique solution a_n and b_n respectively. Moreover, by using lemma 3.1, we have that

$$\alpha = a_0 \leq a_1 \leq \dots \leq a_n \leq b_n \leq \dots \leq b_1 \leq b_0 = \beta \qquad \dots (3.8)$$

on $[-\tau, 2\pi]$. Therefore, sequences $\{a_n(t)\}$ and $\{b_n(t)\}$ are monotonic and bounded.

Hence, for each t

$$\lim_{n \to \infty} a_n(t) = \rho(t) \text{ and } \lim_{n \to \infty} b_n(t) = r(t)$$

monotonically. Furthermore (3.6), (3.7) and (3.8) imply that

$$|a_n'|_{L_i^\infty} \leq K_i \text{ and } |b_n'|_{L_i^\infty} \leq K_i \qquad \dots (3.9)$$

for some constant $K_i > 0$ independent of $n \geq 1$, where $L_i^\infty = L_i^\infty(J_i, R)$ ($i = 1, 2, \dots, m+1, J_1 = [-\tau, t_1], J_2 = [t_1, t_2], \dots, J_i = [t_{i-1}, t_i], \dots, J_{m+1} = [t_m, 2\pi]$) is the usual Lebesque space with the supremes norm. By (3.8), (3.9) and the Arzela-Ascoli theorem, we deduce that $a_n^{(i)} = a_n(t)|_{t \in J_i} \to \rho$ and $b_n^{(i)} = b_n(t)|_t \in J_i \to r^{(i)}$ uniformly on J_i as $n \to \infty$. Hence, we have $a_n \to \rho$ and $b_n \to r$ piecewise uniformly on $[-\tau, 2\pi]$ as $n \to \infty$ where

$$\rho(t) = \begin{cases} \rho^{(1)}(t), t \in [-\tau, t_1] \\ \rho^{(2)}(t), t \in (t_1, t_2] \\ \dots\dots\dots\dots\dots \\ \rho^{(m+1)}(t), t \in (t_m, 2\pi] \end{cases}$$

$$r(t) = \begin{cases} r^{(1)}(t), t \in [-\tau, t_1] \\ r^{(2)}(t), t \in (t_1, t_2] \\ \dots\dots\dots\dots\dots \\ r^{(m+1)}(t), t \in (t_m, 2\pi]. \end{cases}$$

Now, using the uniform convergence, one can easily see that

$$\rho'(t) = f(t, \rho(t), \rho_t),\ t \neq t_k,\ t \in [0, 2\pi]$$

$$\Delta \rho(t_k) = I(\rho(t_k)),\ k = 1, 2, \ldots, m$$

$$\rho(0) = \rho(2\pi)$$

and

$$r'(t) = f(t, r(t), r_t),\ t \neq t_k,\ t \in [0, 2\pi]$$

$$\Delta r(t_k) = I(r(t_k)),\ k = 1, 2, \ldots, m$$

$$r(0) = r(2\pi).$$

Hence, ρ, r are solutions of the PBVP (2.1)-(2.3), and standard argument[10] show that ρ, r are minimal and maximal solution of (2.1)-(2.3) on the interval $[-\tau, 2\pi]$ respectively. The proof is complete.

4. PROOF OF THEOREM 2.2

Lemma 4.1 — Assume that $p \in E$ satisfies

$$p' \leq -Mp - N \int_{t-\tau}^{t} p(s)ds,\ t \neq t_k,\ t \in [0, 2\pi], \quad \ldots (4.1)$$

$$\Delta p(t_k) \leq -L_k\, p(t_k), \quad \ldots (4.2)$$

$$p(t) = p(0),\ t \in [-\tau, 0] \quad \ldots (4.3)$$

and

$$p(0) \leq p(2\pi), \quad \ldots (4.4)$$

where constants $M > 0$, $N \geq 0$, $\tau > 0$, $L_k \in [0, 1)$ $(k = 1, 2, \ldots, m)$ satisfying (2.5). Then $p(t) \leq 0$ for $t \in [0, 2\pi]$.

PROOF : Let $q(t) = p(t)\, e^{Mt}$, $t \in [0, 2\pi]$, then $q(t) \in E$ and

$$q'(t) \leq -N \int_{t-\tau}^{t} q(s)\, e^{M(t-s)}\, ds,\ t \neq t_k,\ t \in [0, 2\pi],\ k = 1, 2, \ldots, m, \quad \ldots (4.5)$$

$$\Delta q(t_k) \leq -L_k\, q(t_k) \quad \ldots (4.6)$$

and

$$q(0) \leq q(2\pi)\, e^{-2\pi M} \quad \ldots (4.7)$$

We now prove $q(t) \leq 0$ on $[0, 2\pi]$. Suppose the conclusion is not true. Then we have two cases (*i*) there exists $t_1^* \in [0, 2\pi]$ such that $q(t_1^*) > 0$, and $q(t) \geq 0$ on $[0, 2\pi]$; (ii) there exist $t_2^*, t_3^* \in [0, 2\pi]$ such that $q(t_2^*) > 0$ and $q(t_3^*) < 0$.

If case (i) holds, then (4.5) implies $q'(t) \le 0$ for $t \in [0, 2\pi], t \ne t_k$ $(k = 1, 2, ..., m)$ and

$$q(t_k^+) \le (1 - L_k)\, q(t_k) \le q(t_k),\ k = 1, 2, ..., m.$$

This means that $q(t)$ is nonincreasing in $[0, 2\pi]$ and therefore

$$q(0) \ge q(t_1^*) > 0,\ q(0) \ge q(2\pi)$$

which contradicts (4.7).

In case (ii), let $\inf_{t \in [0, 2\pi]} q(t) = -\lambda$. Then $\lambda > 0$, and there exists $t_i < t_0^* \le t_{i+1}$ for some i such that $q(t_0^*) = -\lambda$ or $q(t_i^+) = -\lambda$. We may assume that $q(t_0^*) = -\lambda$. In case of $q(t_i^+) = -\lambda$, the proof is similar. From (4.5), it is easy to see that

$$q'(t) \le \lambda N \int_{t-\tau}^{t} e^{M(t-s)}\, ds = \lambda \overline{M},\ t \ne t_k, t \in [0, 2\pi]\ (k = 1, 2, ..., m), \quad \text{... (4.8)}$$

where $\overline{M} = M^{-1} N (e^{\tau} - 1)$, we have

$$q(2\pi) - (t_m^+) = q'(\xi_m)(2\pi - t_m),\ t_m < \xi_m < 2\pi$$

$$q(t_m) - q(t_{m-1}^+) = q'(\xi_{m-1})(t_m - t_{m-1}),\ t_{m-1} < \xi_{m-1} < t_m \quad \text{... (4.9)}$$

....

$$q(t_{i+2}) - q(t_{i+1}^+) = q'(\xi_{i+1})(t_{i+2} - t_{i+1}),\ t_{i+1} < \xi_{i+1} < t_{i+2}$$

$$q(t_{i+1}) - q(t_0^*) = q'(\xi_i)(t_{i+1} - t_0^*),\ t_0^* < \xi_i < t_{i+1}$$

and so, by (4.6) and (4.8)

$$q(2\pi) - (1 - t_m)\, q(t_m) \le \lambda \overline{M}\, \delta$$

$$q(t_m) - (1 - L_{m-1})\, q(t_{m-1}) \le \lambda \overline{M}\, \delta \quad \text{... (4.10)}$$

......

$$q(t_{i+2}) - (1 - L_{i+1})\, q(t_{i+1}) \le \lambda \overline{M}\, \delta$$

$$q(t_{i+1}) + \lambda \le \lambda \overline{M}\, \delta$$

which implies

$$q(2\pi) \le -\lambda \prod_{k=i+1}^{m} (1 - L_k) + \lambda \overline{M}\, \delta \left\{ 1 + \sum_{n=i+1}^{m} \prod_{k=n}^{m} (1 - L_k) \right\}. \quad \text{... (4.11)}$$

If $q(2\pi) > 0$, then (4.11) gives

$$\overline{M}\,\delta > \frac{\prod_{k=i+1}^{m}(1-L_k)}{1+\sum_{n=i+1}^{m}\prod_{k=n}^{m}(1-L_k)} \geq \frac{\prod_{k=1}^{m}(1-L_k)}{1+\sum_{n=1}^{m}\prod_{k=n}^{m}(1-L_k)}$$

which contradicts (2.5). So we have $q(2\pi) \leq 0$, and by (4.7), $q(0) \leq q(2\pi)\,e^{-2\pi M} \leq 0$.

Hence, $0 < t_2^* < 2\pi$. Let $t_j < t_2^* < t_{j+1}$ for some j.

We first assume that $t_0^* < t_2^*$. So, $i \leq j$. We have similar to (4.9)

$$\begin{aligned} &q(t_2^*) - q(t_j^+) = q'(\xi_j)\,(t_2^* - t_j),\ t_j < \xi_j < t_2^* \\ &q(t_j) - q(t_{j-1}^+) = q'(\xi_{j-1})\,(t_j - t_{j-1}),\ t_{j-1} < \xi_{j-1}, t_j \\ &\qquad\vdots \\ &q(t_{i+2}) - q(t_{i+1}^+) = q'(\xi_{i+1})\,(t_{i+2} - t_{i+1}),\ t_{i+1} < \xi_{i+1} < t_{i+2} \\ &q(t_{i+1}) - q(t_0^*) = q'\,\xi_i)\,(t_{i+1} - t_0^*),\ t_0^* < \xi_i < t_{i+1} \end{aligned} \tag{4.12}$$

and so, as in (4.10) and (4.11), we get

$$0 < q(t_2^*) \leq -\lambda \prod_{k=i+1}^{j}(1-L_k) + \lambda \overline{M}\,\delta \left\{ 1 + \sum_{n=j+1}^{j}\prod_{k=n}^{j}(1-L_k) \right\}, \tag{4.13}$$

which implies

$$\overline{M}\,\delta > \frac{\prod_{k=i+1}^{j}(1-L_k)}{1+\sum_{n=i+1}^{j}\prod_{k=n}^{j}(1-L_k)} = \frac{\prod_{k=i+1}^{m}(1-L_k)}{\sum_{k=j+1}^{m}(1-L_k) + \sum_{n=i+1}^{j}\prod_{k=n}^{m}(1-L_k)}$$

$$\geq \frac{\prod_{k=1}^{m}(1-L_k)}{1+\sum_{n=1}^{m}\prod_{k=n}^{m}(1-L_k)}$$

and this contradicts (2.5).

Next assume that $t_2^* < t_0^*$. So $j \leq i$. Similar to (4.12) and (4.13). We have

$$0 < q(t_2^*) \le q(0) \prod_{k=1}^{j} (1 - L_k) + \lambda \overline{M} \delta \left\{ 1 + \sum_{n=1}^{j} \prod_{k=n}^{j} (1 - L_k) \right\},$$

which implies

$$q(0) \prod_{k=1}^{j} (1 - L_k) \ge - \lambda \overline{M} \delta \left\{ 1 + \sum_{n=1}^{j} \prod_{k=n}^{j} (1 - L_k) \right\}. \qquad \text{... (4.14)}$$

It follows from (4.7), (4.11) and (4.14) that

$$- \lambda \overline{M} \delta \left\{ 1 + \sum_{n=1}^{j} \prod_{k=n}^{j} (1 - L_k) \right\}$$

$$< - \lambda e^{-2\pi M} \prod_{k=1}^{j} (1 - L_k) \prod_{k=i+1}^{m} (1 - L_k) + \lambda \overline{M} \delta e^{-2\pi M} \prod_{k=1}^{j} (1 - L_k)$$
$$\left\{ 1 + \sum_{n=i+1}^{m} \prod_{k=n}^{m} (1 - L_k) \right\}$$

or

$$\prod_{k=1}^{j} (1 - L_k) \prod_{k=i+1}^{m} (1 - L_k)$$

$$< \overline{M} \delta e^{2\pi M} \left\{ 1 + \sum_{n=1}^{j} \prod_{k=n}^{j} (1 - L_k) \right\} + \overline{M} \delta \prod_{k=1}^{j} (1 - L_k) \left\{ 1 + \sum_{n=i+1}^{m} \prod_{k=n}^{m} (1 - L_k) \right\}.$$

Hence,

$$\left\{ \prod_{k=1}^{m} (1 - L_k) \right\}^2$$

$$\le \left[\prod_{k=j+1}^{m} (1 - L_k) \right] \left[\prod_{k=1}^{j} (1 - L_k) \prod_{k=i+1}^{m} (1 - L_k) \right]$$

$$< \overline{M} \delta e^{2\pi M} \left\{ \prod_{k=j+1}^{j} (1 - L_k) + \sum_{n=1}^{j} \prod_{k=n}^{m} (1 - L_k) \right\}$$

$$+ \overline{M} \delta \prod_{k=1}^{m} (1 - L_k) \left\{ 1 + \sum_{n=i+1}^{m} \prod_{k=n}^{m} (1 - L_k) \right\}$$

$$\leq \overline{M}\,\delta e^{2\pi M}\left\{1+\sum_{n=1}^{m}\prod_{k=n}^{m}(1-L_k)\right\}+\overline{M}\,\delta\left\{1+\sum_{n=1}^{m}\prod_{k=n}^{m}(1-L_k)\right\}$$

$$=\overline{M}\,\delta(e^{2\pi M}+1)\left\{1+\sum_{n=1}^{m}\prod_{k=n}^{m}(1-L_k)\right\}$$

which contradicts (2.5). The proof is completes.

Let

$$[\alpha,\beta] = \{\eta\in E\cap E_0;\ \alpha\leq\eta\leq\beta\}\ \text{with norm}\ |x|_0=\max\{|x(s)|: s\in[-\tau, 2\pi]\}.$$

We have the following lemma.

Lemma 4.2[10] — Assume that constants $M > 0$, $N \geq 0$, $L_k \in [0, 1)$ $(k = 1, 2, ..., m)$ satisfy the condition (2.6). Then for every $\eta\in[\alpha,\beta]$ the following PBVP of a linear impulsive delay differential equation

$$p'(t)+Mp(t)=\sigma(t)-N\int_{t-\tau}^{t}p(s)ds,\ t\neq t_k,\ t\in[0, 2\pi] \qquad \text{... (4.15)}$$

$$\Delta p(t_k)=-L_k p(t_k)+I_k(\eta(t_k))+L_k\eta(t_k),\ k=1, 2, ..., m \qquad \text{... (4.16)}$$

$$p(t)=p(0),\ t\in[-\tau, 0], \qquad \text{... (4.17)}$$

and

$$p(0)=p(2\pi) \qquad \text{... (4.18)}$$

where

$$\sigma(t)=f(t,\eta(t),\eta_t)+M\eta(t)+N\int_{t-\tau}^{t}\eta(s)\,ds$$

possesses a unique solution.

PROOF OF THEOREM 2.2 — For any $\eta\in[\alpha,\beta]$, we define an operator A by $A\eta=p$, where p is the unique solution of the PBVP (4.15)-(4.18) with

$$\sigma(t)=f(t,\eta(t),\eta_t)+M\eta(t)+N\int_{t-\tau}^{t}\eta(s)ds.$$

Then the operator A has the following properties :

(*i*) $\alpha\leq A\alpha;\ \beta\geq A\beta$

and (*ii*) A is monotone nondecreasing on $[\alpha,\beta]$, i.e., for any

$$\eta_1, \eta_2\in[\alpha,\beta],\ \eta_1\leq\eta_2\ \text{implies}\ A\eta_1\leq A\eta_2.$$

To prove (*i*), set $q = \alpha_0 - \alpha_1$, where $\alpha_1 = A\,\alpha_0$. Then we have

$$q'(t) = \alpha_0'(t) - \alpha_1'(t) \le -Mq(t) - N \int_{t-\tau}^{t} p(s)\,ds,\ t \ne t_k,\ t \in [0, 2\pi]$$

$$\Delta p(t_k) = \Delta \alpha_0(t_n) - \Delta \alpha_1(t_k) \le -L_k q(t_k),\ k = 1, 2, \ldots, m$$

$$q(t) = q(0) \le q(2\pi),\ t \in [-\tau, 0].$$

By lemma 4.1 we get $p(t) \le 0\ (t \in [-\tau, 2\pi])$, i.e., $\alpha \le A\,\alpha$. Similar arguments show that $\beta \ge A\,\beta$. To prove (ii), let $u_1 = A\,\eta_1, u_2 = A\,\eta_2$, where $\eta_1 \le \eta_2$ on $[-\tau, 2\pi]$ and $\eta_1, \eta_2 \in [\alpha, \beta]$. Set $q = q_1 - q_2$ we get

$$q'(t) = u_1'(t) - u_2'(t) \le -Mq(t) - N \int_{t-\tau}^{t} q(s)ds,\ t \ne t_k,\ t \in [0, 2\pi]$$

$$\Delta q(t_k) = \Delta \mu_1(t_k) - \Delta \mu_2(t_k) \le -L_k q(t_k),\ k = 1, 2, \ldots, m$$

$$q(t) \equiv q(0) = p(2\pi),\ t \in [-\tau, 0]$$

In view of lemma 4.1, we have $q(t) \le 0\ (-\tau \le t \le 2\pi)$. Define the sequences of functions $\{a_n(t)\}$ and $\{b_n(t)\}$ by the equalities

$$a_0 = \alpha,\ \beta_0 = \beta\ a_{n+1} = Aa_n,\ b_{n+1} = Ab_n.$$

From (i) and (ii), we obtain

$$a_0 \le a_1 \le a_2 \le \ldots \le a_n \le \ldots \le b_n \le \ldots \le b_2 \le b_1 \le b_0, \text{ on } [-\tau, 2\pi].$$

Each $a_n, b_n \in E \cap E_0$, $n = 1, 2, \ldots$, satisfies the PBVP

$$a_n'(t) + Ma_n(t) = f(t, a_{n-1}(t), a_{n-1_t}) + Ma_{n-1}(t)$$

$$-N \int_{t-\tau}^{t} [a_n(s) - a_{n-1}(s)]\,ds,\ t \ne t_k,\ t \in [0, 2\pi],$$

$$\Delta a_n(t_k) = -L_k a_n(t_k) + I_k(a_{n-1}(t_k)) + L_k a_{n-1}(t_k),\ k = 1, 2, \ldots, m$$

$$a_n(0) = a_n(2\pi)$$

and

$$b_n'(t) + Mb_n(t) = f(t, b_{n-1}(t), b_{n-1_t}) + Mb_{n-1}(t)$$

$$- \int_{t-\tau}^{t} [b_n(s) - b_{n-1}(s)]\, ds,\ t \neq t_k,\ t \in [0, \pi],$$

$$\Delta b_n(t_k) = -L_k b_n(t_k) + I_k(b_{n-1}(t_k)) + L_k b_{n-1}(t_k),\ k = 1, 2, \ldots, m$$

$$b_n(0) = b_n(2\pi).$$

Therefore, there exists ρ, r such that $\lim_{n \to \infty} a_n(t) = \rho(t)$, $\lim_{n \to \infty} b_n(t) = r(t)$ piecewise uniformly on $[-\tau, 2\pi]$. Clearly, ρ, r satisfy the PBVP (2.1)-(2.3) (2.7) and standard argument[10] show that ρ, r are minimal and maximal solution of (2.1)-(2.3) (2.7) on the $[-\tau, 2\pi]$. The proof of the theorem is complete.

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