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# VARIATIONAL LYAPUNOV METHOD AND STABILITY THEORY*

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In this paper, by using variational Lyapunov method, we will establish some criteria on the properties of solutions of the perturbed differential equation, such as stability, boundedness. Compared with the former results obtained by using Lyapunov method, it is not difficult to find that variational Lyapunov method is an extension of Lyapunov method and the former results can be viewed as the special cases of the theorems here.

**Key Words : Variational Lyapunov Method; Strict Stability; Strict Boundedness**

## 1. INTRODUCTION

It is well known that the formula of variation of parameters and the method of Lyapunov functions are both extremely useful and effective techniques in the study of the effect of perturbations of differential equations. These methods, however, indicate that we measure the perturbation by means of a norm and thus destroy the good nature, if any, of the perturbing terms. Recently a new method-variational Lyapunov method was developed to correct this unpleasant situation, see[1&5] for example.

In this paper, we are concerned with stability and boundedness of solutions of the perturbed differential system. We will show that because of the effect of perturbations, solutions of the perturbed differential system have stronger properties than those of its corresponding nonperturbed differential system. In order to obtain such solutions, we will use variational Lyapunov method.

This paper is organized as follows: in Section 2 we make preparations for the following sections. Section 3 and section 4 deal with stability and boundedness of solutions respectively.

## 2. PRELIMINARIES

We consider the two differential systems

$$y' = f(t, y), y(t_0) = x_0, \quad \text{... (2.1)}$$

and

$$x' = F(t, x), x(t_0) = x_0, \quad \text{... (2.2)}$$

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where $f, F \in C[R_+ \times S(p), R^n]$. Here $S(p) = \{x \in R^n : \| x \| < p\}$, where $\| \cdot \|$ is a norm in $R^n$. Throughout this paper, relative to the system (2.1) assume that

($H$) the solutions $y(t) = y(t, t_0, x_0)$ of (2.1) exist for all $t \geq t_0$, are unique, continuous with respect to the initial data and $\| y(t, t_0, x_0) \|$ is locally Lipschitzian in $x_0$. Since ($H$) implies that $\| y(t, s, x) \| < p$ for $t \geq s$, $x \in S(p)$, for any $V \in C[R_+ \times S(p), R_+^N]$ and any fixed $t \in (t_0, \infty)$, we introduce variational Lyapunov function $V(s, y(t, s, x))$ and its Dini derivative

$$D^+ V(s, y(t, s, x)) \equiv \limsup_{h \to 0^+} \frac{1}{h} [V(s+h, y(t, s+h, x+hF(s, x))) - V(s, y(t, s, x))] \quad \text{... (2.3)}$$

for $t_0 \leq s \leq t$ and $x \in S(p)$.

The following lemma which relates the solutions of (2.2) to the solutions of (2.1) is an important tool for our discussion.

*Lemma* 2.1[1] — Assume that

(1) $V \in C[R_+ \times S(p), R_+^N)$, $V(t, x)$ is locally Lipschitzian in $x$ and $t_0 \leq s \leq t, x \in S(p)$,

$$D^+ V(s, y(t, s, x)) \leq g(t, s, V(s, y(t, s, x))), \quad \text{... (2.4)}$$

(2) $g \in C[R_+^2 \times R_+^N, R^N]$, $g(t, s, u)$ is quasimonotone nondecreasing in $u$ for each $(t, s)$ and $r(t, s, t_0, u_0)$ is the maximal solution of

$$\frac{du}{ds} = g(t, s, u), \; u(t_0) = u_0 \geq 0, \quad \text{... (2.5)}$$

existing for $t_0 \leq s \leq t < \infty$.

Then $V(t_0, y(t, t_0, x_0)) \leq u_0$ implies

$$V(t, x(t, t_0, x_0)) \leq r_0(t, t_0, u_0), \; t \geq t_0, \quad \text{... (2.6)}$$

where $r_0(t, t_0, u_0) \equiv r(t, t, t_0, u_0)$, $x(t) = x(t, t_0, x_0)$ is any solution of (2.2).

*Remark* 1 : Taking $u_0 = V(t_0, y(t, t_0, x_0))$, the inequality (2.6) becomes

$$V(t, x(t, t_0, x_0)) \leq r_0(t, t_0, V(t_0, y(t, t_0, x_0)), \; t \geq t_0,$$

which shows the connection between the solutions of (2.1) and (2.2) in terms of the maximal solution of (2.5).

*Remark* 2 : Suppose that $g(t, s, u) = g(s, u)$, then the system (2.5) becomes

$$\frac{du}{ds} = g(s, u), \; u(t_0) = u_0 \geq 0,$$

where $r(t, t_0, u_0)$ is the maximal solution.

*Remark* 3 : If $f(t, y) \equiv 0$, then $y(t, t_0, x_0) \equiv x_0$ and thus the hypothesis $(H)$ is trivially verified. Since $y(t, s, x) \equiv x$, the definition (2.3) reduces to

$$D^+ V(s, x) \equiv \limsup_{h \to 0^+} \frac{1}{h} [V(s+h, x + F(s, x)) - V(s, x)],$$

which is the usual definition of the derivative of vector Lyapunov functions relative to (2.2). Consequently the usual comparison theorem in [2, 3, 4] is imbedded as a special case in our present result.

Lyapunov stability of the invariant set of a differential system does not rule out the possibility of asymptotic stability of the set. Various definitions of stability and boundedness are one-sided estimates and thus they are not strict concepts in a sense. It is natural to expect that an estimation of the lower bound for the rate at which the solutions approach the invariant set would yield interesting refinements for stability notions. We introduce below the concepts of strict stability and boundedness of solutions.

*Definition* 2.1 — The trivial solution of (2.1) is said to be

(1) strictly stable, if for any $\varepsilon_1 > 0, t_0 \in R_+$, there exists a $\delta_1 = \delta_1 (t_0, \varepsilon_1) > 0$ such that $\| x_0 \| \leq \delta_1$ implies $\| y(t, t_0, x_0) \| < \varepsilon_1, t \geq t_0$. Moreover for any $\delta_2 \in (0, \delta_1]$, there exists a $\varepsilon_2 = \varepsilon_2 (t_0, \delta_2) > 0$ such that $\| x_0 \| \geq \delta_2$ implies $\| y (t, t_0, x_0) \| > \varepsilon_2, t \geq t_0$.

(2) strictly uniformly stable, if $\delta_1$, $\delta_2$ and $\varepsilon_2$ are independent of $t_0$.

*Definition* 2.2 — The system (2.1) is said to be

(1) strictly bounded, if for any $\alpha_1 \geq \alpha_2 > 0, t_0 \in R_+$, there exist $\beta_1 = \beta_1 (t_0, \alpha_1) > 0, \beta_2 = \beta_2 (t_0, \alpha_2) > 0$ such that $\alpha_2 \leq \| x_0 \| \leq \alpha_1$ implies $\beta_2 < \| y(t, t_0, x_0) \| < \beta_1, t \geq t_0$.

(2) strictly uniformly bounded, if $\beta_1, \beta_2$ are independent of $t_0$.

Let us define the following classes of functions for the future use.

$$K = \{a \in C [R_+, R_+] : a(u) \text{ is strictly increasing in } u \text{ and } a(0) = 0\},$$

$$K' = \{a \in C [R_+, R_+] : a(u) \text{ is nondecreasing in } u\}.$$

## 3. Main Results of Stability

We shall consider in this section some criteria on stability and asymptotic behaviour of solution of (2.2). Throughout this section assume that $f(t, 0) \equiv 0, F(t, 0) \equiv 0$.

***Theorem*** 3.1 — *Assume that*

*(1)* $V \in C [R_+ \times S(p), R_+]$, $V(t, x)$ *is loclly Lipschitzian in* $x$*; and*

*(2) there exist two functions* $a, b \in K$ *such that*
$b(\| x \|) \leq V(t, x) \leq a(\| x \|), (t, x) \in R_+ \times S(p)$;

*(3)* $D^+ V (s, y(t, s, x)) \leq 0$, *for* $s \leq t, (s, x) \in R_+ \times S(p)$; *and*

*(4) The trivial solution of (2.1) is uniformly stable.*

*Then the trivial solution of (2.2) is uniformly stable.*

PROOF : Let $0<\varepsilon<p, t_0\in R_+, \delta_1=\delta_1(\varepsilon)>0$ be given such that $a(\delta_1)<b(\varepsilon)$. Since $y = 0$ of (2.1) is uniformly stable, given $\delta_1>0, t_0\in R_+$, there exists a $\delta=\delta(\varepsilon)>0$ such that

$$\| y(t, t_0, x_0)\| < \delta_1, t\geq t_0 \text{ provided that } \| x_0\| \leq \delta \qquad \text{... (3.1)}$$

We claim that $\| x_0\| \leq \delta$ also implies $\| x(t, t_0, x_0)\| < \varepsilon, t\geq t_0$, where $x(t, t_0, x_0)$ is any solution of (2.2). If this is not true, there would exist a solution $x(t)=x(t, t_0, x_0)$ of (2.2) with $\| x_0\| \leq \delta$ and $t_1>t_0$ such that

$$\| x(t_1)\| = \varepsilon, \text{ and } \| x(t)\| < \varepsilon, t\in [t_0, t_1).$$

Then by lemma 2.1, we have

$$V(t, x(t)) \leq V(t_0, y(t, t_0, x_0)), t\in [t_0, t_1].$$

Consequently, in view of (3.1) and (2), we get

$$V(t_1, x(t_1)) \leq V(t_0, y(t_1, t_0, x_0)) \leq a(\| y(t_1, t_0, x_0)\|) < a(\delta_1) < b(\varepsilon),$$

which contradicts the fact that $V(t_1, x(t_1)) \geq b(\| x(t_1)\|) = b(\varepsilon)$. Thus we prove that $x = 0$ of (2.2) is uniformly stable.

*Remark* 1 : It is not necessary to find a suitable known system (2.1). We only choose a function $y\in C[R_+^2\times R^n, R^n]$ satisfying (i) $\| y(t, t_0, x_0)\|$ is locally Lipschitzian in $x_0$ for each $(t, t_0)$; (ii) $y(t_0, t_0, x_0)=x_0$; (iii) $y(t, t_0, x_0)$ has required stability properties.

*Remark* 2 : The former Lyapunov original theorem is the special case of theorem here. In this case, we only take $y(t, t_0, x_0)=x_0$.

The following result is about the unstability of $x = 0$ of (2.2).

***Theorem*** 3.2 — *Assume that there exists a $h > 0$ and a $\tau\in R_+$ such that $S(h)\subset S(p)$. A function $V: [\tau, \infty)\times S(h)\to R$ is continuous and locally Lipschitzian in $x$, $V(t, 0)\equiv 0$.*

*Moreover,*

*(1) For any $\delta>0, t_0\geq\tau$, there exists a $x_0\in S(\delta)$ such that $V(t_0, x_0)>0$;*

*(2) there exists a function $a\in K$ such that*

$$V(t, x)\leq a(\| x\|), (t, x)\in [\tau, \infty)\times S(h);$$

*(3) there exists a function $c\in K$ such that*

$$D^+ V(s, y(t, s, x)) \geq c(\| y(t, s, x)\|), (s, x)\in [\tau, \infty)\times S(h);$$

*(4) the set $S(h)$ is the invariant set of (2.1), i.e., for $x_0\in S(h), t_0\in R_+, y(t, t_0, x_0) \in S(h), t\geq t_0$, where $y(t, t_0, x_0)$ is any solution of (2.1); and*

*(5) the trivial solution of (2.1) is strictly uniformly stable and if $V(t_0, x_0)>0$, then there exist a $\eta>0, \alpha>0$ and a sequence of intervals $[\alpha_i, \beta_i], \alpha_i<\beta_i<\alpha_{i+1}, \beta_i-\alpha_i\geq\alpha$, such that*

$$V(t_0, y(t, t_0, x_0)) \geq \eta,\ t \in [\alpha_i, \beta_i],\ i = 1, 2, \ldots$$

*Then the trivial solution of (2.2) is unstable.*

PROOF : In view of (1), for any $\delta > 0, t_0 \geq \tau$, there exists a $x_0 \in S(\delta)$ such that $V(t_0, x_0) > 0$. Thus together with (5), we have $V(t_0, y(t, t_0, x_0)) \geq \eta$, $t \in [\alpha_i, \beta_i]$, $i = 1.2$ .... Obviously to draw the conclusion we only prove that the solution $x(t) = x(t, t_0, x_0)$ of (2.2) will leave $S(h)$ at some time. In fact, if this is not true, then $x(t) \in S(h), t \geq t_0$ and thus by (4)

$$y(t, s, x(s)) \in S(h), t \geq s \geq t_0.$$

Since $D^+ V(s, y(t, s, x)) \geq 0$, we can easily conclude $V(s, y(t, s, x(s)))$ is nondecreasing in $s$ and

$$V(t, x(t)) \geq V(t_0, y(t, t_0, x_0)), t \geq t_0,$$

especially for $t \in [\alpha_i, \beta_i]$, $V(t, x(t)) \geq \eta$. Furthermore in view of (2), we have

$$\| x(t) \| \geq a^{-1} (\eta) > 0, t \in [\alpha_i, \beta_i],\ i = 1.2 \ldots$$

The strict uniform stability of $y = 0$ of (2.1) implies that there exists a $\varepsilon_2 > 0$ such that

$$\| y(t, s, x(s)) \| \geq \varepsilon_2, t \geq s, s \in [\alpha_i, \beta_i],\ i = 1.2 \ldots$$

For the convenience let $x(\alpha_i) = x_{i1}, x(\beta_i) = x_{i2}$. On the other hand because of the fact that $V(s, y(t, s, x(s)))$ is nondecreasing in $s$ and (3), we have for $t \in [\alpha_{k+1}, \beta_{k+1}]$,

$$a(h) \geq a(\| x(t) \|) \geq V(t, x(t)) \geq V(\beta_k, y(t, \beta_k, x_{k2}))$$

$$\geq V(\alpha_k, y(t, \alpha_k, x_{k1})) + \int_{\alpha_{\alpha_k}}^{\beta_k} D^+ V(s, y(t, s, x))\, ds$$

$$\geq V(\alpha_k, y(t, \alpha_k, x_{k1})) + c(\varepsilon_2)(\beta_k - \alpha_k)$$

$$\geq V(\beta_{k-1}, y(t, \beta_{k-1}, x_{k-1,2})) + c(\varepsilon_2)(\beta_k - \alpha_k)$$

$$\geq \ldots$$

$$\geq V(t_0, y(t, t_0, x_0)) + c(\varepsilon_2) \sum_{i=1}^{k} (\beta_i - \alpha_i)$$

$$\geq V(t_0, y(t, t_0, x_0)) + c(\varepsilon_2)\, \alpha k \geq \eta + c(\varepsilon_2)\, \alpha k \to \infty$$

This contradiction that we lead to proves the conclusion.

*Remark* : The conditions of (4) and (5) hold obviously in the following two cases :

(1) Taking $y(t, t_0, x_0) \equiv x_0$, it is an extension of the known unstable theorem;

(2) In two dimensions, take $y(t, t_0, x_0) = (y_1(t, t_0, x_0), y_2(t, t_0, x_0))$ with the initial value $x_0 = (x_{01}, x_{02})$, where

$$\begin{cases} y_1(t, t_0, x_0) = x_{01} \cos(t - t_0) - x_{02} \sin(t - t_0), \\ y_2(t, t_0, x_0) = x_{01} \sin(t - t_0) + x_{02} \cos(t - t_0). \end{cases}$$

From the above discussion, we can clearly see the superiority of variational Lyapunov method. First of all, it is an extension of Lyapunov method and thus the known results[2, 3, 4, 5] are the special cases. Second, because of the perturbation, we can deduce that the weaker stability of $y = 0$ of (2.1) implies $x = 0$ of (2.2) has a stronger stability, which is manifest in the theorem below.

***Theorem* 3.3** — *Assume that*

*(1)* $V \in C[R_+ \times S(p), r_+]$, $V(t, x)$ *is locally Lipschitzian in* $x$;

*(2) there exist two functions* $a, b \in K$ *such that*

$$b(\| x \|) \le V(t, x) \le a(\| x \|), \ (t, x) \in R_+ \times S(p);$$

*(3) there exists a function* $c \in K$ *such that*

$$D^+V(s, y(s, x)) \le -c(\| y(t, s, x) \|), \ s \le t, \ (s, x) \in R_+ \times S(p); \text{ and}$$

*(4) the trivial solution of (2.1) is strictly uniformly stable.*

*Then the trivial solution of (2.2) is uniformly asymptotically stable.*

PROOF : By Theorem 3.1, $x = 0$ of (2.2) is uniformly stable, thus for any $\varepsilon \in (0, p)$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\| x_0 \| \le \delta$ implies $\| x(t, t_0, x_0) \| < \varepsilon$, $t \ge t_0$. Together with the fact that $y = 0$ of (2.1) is uniformly stable, given $p > 0$, there exist a $\delta_0 > 0$ such that $\| x(t, t_0, x_0) \| < p$, $\| y(t, t_0, x_0) \| < p$, $t \ge t_0$ provided that $\| x_0 \| \le \delta_0$. By (4), for the above $\delta > 0$, there exists a $\varepsilon_2 > 0$ such that $\| x_0 \| \ge \delta$ implies $\| y(t, t_0, x_0) \| > \varepsilon_2$, $t \ge t_0$. Let $T = T(\varepsilon) = \dfrac{a(p) + 1}{c(\varepsilon_2)}$, then the conclusion holds for the above $\delta_0, T$, i.e., $\| x(t, t_0, x_0) \| < \varepsilon$, $t \ge t_0 + T$ provided that $\| x_0 \| \le \delta_0$. To complete the proof, we only prove that there exists a $t^* \in [t_0, t_0 + T]$ such that $\| x(t^*, t_0, x_0) \| \le \delta$. In fact, if this is not true, there would exist a solution $x(t) = x(t, t_0, x_0)$ of (2.2) such that for all $t \in [t_0, t_0 + T]$, we have $\| x(t) \| > \delta$. Hence

$$\| y(t, s, x(s) \| \ge \varepsilon_2, \ t \ge s, \ s \in [t_0, t_0 + T].$$

In view of (3), for $t \in [t_0, t_0 + T]$ we get

$$V(t, x(t)) \le V(t_0, y(t, t_0, x_0)) - \int_{t_0}^{t} c(\| y(t, s, x(s)) \|) \, ds,$$

especially when $t = t_0 + T$,

$$0 \le V(t_0 + T, x(t_0 + T)) \le V(t_0, y(t_0 + T, t_0, x_0)) - \int_{t_0}^{t_0+T} c(\| y(t_0 + T, s, x(s)) \|)\, ds$$

$$\le a(\| y(t_0 + T, t_0, x_0) \|) - c(\varepsilon_2)\, T$$

$$\le a(p) - c(\varepsilon_2) \frac{a(p)+1}{c(\varepsilon_2)} < 0.$$

This is a contradiction and thus we complete the proof.

As an application, we shall present two examples.

*Example* 3.1 — Consider the two differential systems

$$y' = f(t) y^2,\ y(t_0) = x_0 > 0, \qquad \text{... (3.1)}$$

$$x' = f(t)\, x^2 - Cx^2,\ \ x(t_0) = x_0 > 0, \qquad \text{... (3.2)}$$

where $f \in C[R_+, R_+]$, $0 < \int_0^\infty f(t)\, dt < \infty$, $C$ is an arbitrary positive constant.

Solve (3.1)

$$y(t) = y(t, t_0, x_0) = \frac{x_0}{1 - x_0 \int_{t_0}^{t} f(s) ds}.$$

Since $y' \ge 0$,

$$x_0 \le y(t) \le \lim_{t \to \infty} y(t) = \frac{x_0}{1 - x_0 \int_{t_0}^{\infty} f(s) ds} < \frac{x_0}{1 - x_0 \int_{0}^{\infty} f(s) ds}.$$

Thus for $0 < x_0 < \dfrac{1}{\int_0^\infty f(s) ds}$, $y = 0$ of (3.1) is strictly uniformly stable.

Consider the variational problem

$$Z' = 2f(t)\frac{x_0}{1 - x_0 \int_{t_0}^{t} f(s)ds} Z,\ Z(t_0) = 1,$$

where $$Z(t) = \frac{1}{\left(1 - x_0 \int_{t_0}^{t} f(s)ds\right)^2}$$ is the solution, hence

$$\frac{\partial y(t, t_0, x_0)}{\partial x_0} = \frac{1}{\left(1 - x_0 \int_{t_0}^{t} f(s)ds\right)^2}.$$

Let $V(x) = |x|^2$, then

$$D^+ V = 2y(t, s, x)\frac{\partial y(t, s, x)}{\partial x} R(s, x) = -2C|y(t, s, x)|^3.$$

Moreover $y(t, t_0, x_0)$ is locally Lipschitzian in $x_0$. In fact, for $0 < x_0 \le \frac{1}{2\int_0^{\infty} f(s)ds}$, $t \ge t_0$, we have

$$0 < y'_{x_0} = \frac{1}{\left(1 - x_0 \int_{t_0}^{t} f(s)\,ds\right)^2} \le \frac{1}{\left(1 - x_0 \int_{0}^{\infty} f(s)\,ds\right)^2} \le C',$$

where $C'$ is positive. By Theorem 3.3, $x = 0$ of (3.2) is uniformly asymptotically stable.

*Remark* : There two special cases of $f(t)$ to discuss:

(1) $f(t) = e^{-t}$, $C = \frac{1}{2}$; and

(2) $f(t) = \frac{1}{(t+1)^2}$, $C = \frac{1}{2}$.

*Example* 3.2 — Consider differential equations

$$x_1' = -x_2 + \alpha x_1^3,\ x_2 = x_1 + \alpha x_2^3, \qquad \text{... (3.3)}$$

where $x_1 = x_2 = 0$ is its solution.

Consider its corresponding nonperturbed differential equations

$$x_1' = -x_2, \; x_2' = x_1, \qquad \text{... (3.4)}$$

which has a solution $y(t, t_0, x_0) = (y_1(t, t_0, x_0), y_2(t, t_0, x_0))$ with the initial value $x_0 = (x_{01}, x_{02})$, where

$$\begin{cases} y_1(t, t_0, x_0) = x_{01}\cos(t - t_0) - x_{02}\sin(t - t_0), \\ y_2(t, t_0, x_0) = x_{01}\sin(t - t_0) + x_{02}\cos(t - t_0). \end{cases}$$

Obviously the trivial solution of (3.4) is strictly uniformly stable.

Let $\quad V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, then

$$\frac{dV}{ds} = (y_1(t, s, x(s)), y_2(t, s, x(s)))\begin{pmatrix} \cos(t-s) & -\sin(t-s) \\ \sin(t-s) & \cos(t-s) \end{pmatrix}\begin{pmatrix} \alpha x_1^3(s) \\ \alpha x_2^3(s) \end{pmatrix}$$

$$= \alpha\left(x_1^4(s) + x_2^4(s)\right)$$

If $\alpha < 0$, then $\frac{dV}{ds} \leq \frac{\alpha}{2}(x_1^2(s) + x_2^2(s))^2 = \frac{\alpha}{2}|y(t, s, x(s))|^4$, and thus $x_1 = x_2 = 0$ of (3.3) is uniformly asymptotically stable.

If $\alpha > 0$, then $\frac{dV}{ds} \geq \frac{\alpha}{2}(x_1^2(s) + x_2^2(s))^2 = \frac{\alpha}{2}|y(t, s, x(s))|^4$, and thus $x_1 = x_2 = 0$ of (3.3) is unstable.

If $\alpha = 0$, then (3.3) becomes (3.4) and thus $x_1 = x_2 = 0$ is strictly uniformly stable.

In short, from the above theorems we can easily deduce that variational Lyapunov method is an extension of Lyapunov method. Moreover, one can follow the proofs of other stability results in[2, 3, 4] with appropriate modifications and obtain similar results by using variational Lyapunov method. Thus details are omitted.

## 4. Main Results of Boundedness

In this section, we will discuss the boundedness and attractivity properties of (2.2).

***Theorem* 4.1 — *Assume that***

*(1)* $V \in C[R_+ \times S^c(\eta), R_+]$, $V(t, x)$ *is locally Lipschitzian in* $x$;

*(2) there exist two functions* $a, b \in K'$ *with* $\lim_{r \to \infty} b(r) = \infty$ *such that*

$$b(\|x\|) \leq V(t, x) \leq a(\|x\|), \; (t, x) \in R_+ \times S^c(\eta);$$

*(3)* $D^+V(s, y(t, s, x)) \leq 0$, $(s, x) \in R_+ \times S^c(\eta)$, $s \leq t$; *and*

*(4) the system (2.1) is strictly uniformly bounded.*

*Then the system (2.2) is uniformly bounded, where* $\eta$ *is determined as follows : for given* $\rho > 0$, *there exists a* $\eta = \eta(\rho) > 0$ *such that* $\|x_0\| \geq \rho$ *implies that* $\|y(t, t_0, x_0)\| > \eta$, $t \geq t_0$.

PROOF : By (4), for any $\alpha > \rho > 0$, there exists a $\beta_1 = \beta_1(\alpha) > 0$ such that $\| x_0 \| \le \alpha$ implies that $\| y(t, t_0, x_0) \| < \beta_1, t \ge t_0$. Let $\beta = \beta(\alpha) > 0$ such that $b(\beta) > a(\beta_1)$. Then the conclusion holds with the choice of $\beta > 0$, i.e., $\| x(t, t_0, x_0) \| < \beta, t \ge t_0$ provided that $\| x_0 \| \le \alpha$. In fact, if this is not true, there would exist a solution $x(t) = x(t, t_0, x_0)$ of (2.2) and $t_2 > t_1 \ge t_0$ such that

$$\| x(t_1) \| = \alpha, \| x(t_2) \| = \beta,$$

and
$$\alpha \le \| x(t) \| \le \beta, t \in [t_1, t_2].$$

In view of (3), we have

$$V(t, x(t)) \le V(t_1, y(t, t_1, x(t_1))), t \in [t_1, t_2],$$

especially when $t = t_2$,

$$V(t_2, x(t_2)) \le V(t_1, y(t_2, t_1, x(t_1))) \le a(\| y(t_2, t_1, x(t_1)) \|) \le a(\beta_1).$$

This contradicts the fact that $V(t_2, x(t_2)) \ge b(\| x(t_2) \|) = b(\beta) > a(\beta_1)$ and thus the system (2.2) is uniformly bounded.

***Theorem*** 4.2 — *Assume that*

*(1)* $V \in C[R_+ \times S^c(\eta), R_+]$, $V(t, x)$ *is locally Lipschitzian in* $x$;

*(2) there exist two functions* $a, b \in K'$ *with* $\lim_{r \to \infty} b(r) = \infty$ *such that*

$$b(\| x \|) \le V(t, x) \le a(\| x \|), (t, x) \in R_+ \times S^c(\eta);$$

*(3) there exists a function* $c \in K$ *such that*

$$D^+ V(s, y(t, s, x)) \le -c(\| y(t, s, x) \|), (s, x) \in R_+ \times S^c(\eta), s \le t; \text{ and}$$

*(4) the system (2.1) is strictly uniformly bounded.*

*Then the system (2.2) is ultimately uniformly bounded, where* $\eta$ *is determined as the above.*

PROOF : By Theorem 4.1, the system (2.2) is uniformly bounded, i.e., for given $\rho > 0$, there exists a $\beta_0 = \beta_0(\rho) > 0$ such that $\| x_0 \| \le \rho$ implies that $\| x(t, t_0, x_0) \| < \beta_0, t \ge t_0$. Since the system (2.1) is uniformly bounded, for any $\alpha > \rho > 0$, there exists a $\beta = \beta(\alpha) > 0$ such that $\| x_0 \| \le \alpha$ implies $\| y(t, t_0, x_0) \| < \beta, t \ge t_0$. Let $T = T(\alpha) = \dfrac{a(\beta) + 1}{c(\eta)}$, then the conclusion holds with the choice of $\beta_0$, $T$, i.e., $\| x(t, t_0, x_0) \| < \beta_0, t \ge t_0 + T$ provided that $\| x_0 \| \le \alpha$. To draw the conclusion we only prove that there exists a $t^* \in [t_0, t_0 + T]$ such that $\| x(t^*, t_0, x_0) \| \le \rho$. In fact, if this is not true, there would exist a solution $x(t) = x(t, t_0, x_0)$ of (2.2) such that for all $t \in [t_0, t_0 + T]$ we have $\| x(t) \| > \rho$, thus $\| y(t, s, x(s)) \| > \eta, t \ge s, s \in [t_0, t_0 + T]$. In view of (3), for $t \in [t_0, t_0 + T]$ we get

$$V(t, x(t)) \leq V(t_0, y(t, t_0, x_0)) - \int_{t_0}^{t} c(\| y(t, s, x(s)) \|)\, ds,$$

especially when $t = t_0 + T$,

$$0 \leq V(t_0 + T, x(t_0 + T)) \leq V(t_0, y(t_0 + T, t_0, x_0)) - \int_{t_0}^{t_0+T} c(\| y(t_0 + T, s, x(s)) \|)\, ds$$

$$\leq a(\| y(t_0 + T, t_0, x_0) \|) - c(\eta) T$$

$$\leq a(\beta) - c(\eta) \frac{a(\beta) + 1}{c(\eta)} < 0.$$

This is a contradiction and thus we complete the proof.

*Remark* : From the above two theorems, we can easily see that the former results, for example, [2, 3, 4] and the references therein, are the special cases.

***Theorem*** 4.3 — *Assume that*

*(1)* $V \in C[R_+ \times R^n, R_+]$, *$V(t, x)$ is locally Lipschitzian in x;*

*(2) there exists a function* $b \in K'$ *with* $\lim_{r \to \infty} b(r) = \infty$ *such that*

$$V(t, x) \geq b(\| x \|), (t, x) \in R_+ \times R^n;$$

*(3) there exists a function* $c \in K$ *such that*

$$D^+ V(s, y(t, s, x)) \leq -c(\| y(t, s, x) \|), s \leq t;$$

*(4) there exists a* $H \in C[R_+, R]$ *satisfying* $H(y) = 0 \Leftrightarrow y = 0$ *and* $\int_0^t [D^+ H(\| x(s) \|)]_\pm\, ds$ *is uniformly continuous in* $R_+$, *where* $[\ ]_\pm$ *denote positive and negative parts,* $x(s)$ *is a continuous function defined in* $R_+$, *and*

*(5) the system (2.1) is strictly uniformly bounded.*

*Then the system (2.2) is equi-bounded and attractive in the large scale.*

PROOF : The proof that the system (2.2) is equi-bounded is immediate, so we omit details. We only prove that the system (2.2) is attractive in the large scale, i.e., for any $\alpha > 0$, $\| x_0 \| \leq \alpha$, $\lim_{t \to \infty} \| x(t, t_0, x_0) \| = 0$. Since the system (2.1) is uniformly bounded, there exists a $\beta_1 = \beta_1(\alpha) > 0$ such that $\| x_0 \| \leq \alpha$ implies that $\| y(t, t_0, x_0) \| < \beta_1, t \geq t_0$. First we claim $\liminf_{t \to \infty} \| x(t, t_0, x_0) \| = 0$. In fact, if this is not true, there would exist a solution $x(t) = x(t, t_0, x_0)$ of (2.2) and $\eta > 0, T \geq t_0$ such that $\| x(t) \| \geq \eta, t \geq T$. In view of (5), for the above $\eta > 0$, there exists a $\beta_2 > 0$ such that $\| x(s) \| \geq \eta$ implies that $\| y(t, s, x(s)) \| > b\eta_2, t \geq s$. Thus together with (3)

$$0 \le V(t, x(t)) = \le V(t_0, y(t, t_0, x_0)) - \int_{t_0}^{t} c(\| y(t, s, x(s)) \|)\, ds$$

$$\le V(t_0, y(t, t_0, x_0)) - \int_{T}^{t} c(\| y(t, s, x(s)) \|)\, ds$$

$$\le \max_{\| y \| \le \beta_1} V(t_0, y) - c(\beta_2)(t - T) \to -\infty,\ t \to \infty.$$

This is a contradiction and thus we prove that $\liminf_{t \to \infty} \| x(t, t_0, x_0) \| = 0$.

Next we claim $\limsup_{t \to \infty} \| x(t, t_0, x_0) \| = 0$. In fact, if this is not true, there would exist a solution $x(t) = x(t, t_0, x_0)$ of (2.2) and $\eta > 0$, $\{t_n\}$, $t_n \to \infty, n \to \infty$ such that $\| x(t_n) \| \ge \eta$. Since $H(y) = 0 \Leftrightarrow y = 0$ and $H$ is continuous, there exist $\sigma, \upsilon > 0$ such that $y \ge \eta$ implies $| H(y) | > \sigma$, and $0 < y < \upsilon < \eta$ implies $| H(y) | < \frac{\sigma}{2}$. Notice that $\liminf_{t \to \infty} \| x(t) \| = 0$, there exist sequences $\{t_n^{(1)}\}$, $\{t_n^{(2)}\}$ and $t_n^{(i)} \to \infty$, $i = 1.2$, satisfying $t_1^{(1)} < t_1^{(2)} < t_2^{(1)} < t_2^{(2)} < \ldots < t_n^{(1)} < t_n^{(2)} < \ldots$ and

$$H(\| x(t_n^{(1)}) \|) = \frac{\sigma}{2},\ H(\| x(t_n^{(2)}) \|) = \frac{3\sigma}{4}$$

or
$$H(\| x(t_n^{(1)}) \|) = -\frac{\sigma}{2},\ H(\| x(t_n^{(2)}) \|) = -\frac{3\sigma}{4}.$$

For $[t_n^{(1)}, t_n^{(2)}]$, we have $\frac{\sigma}{2} \le | H(\| x(t) \|) | \le \frac{3\sigma}{4}$, $n = 1.2\ \ldots$

Thus
$$\frac{\sigma}{4} = H(\| x(t_n^{(2)}) \|) - H(\| x(t_n^{(1)}) \|) \le \int_{t_n^{(1)}}^{t_n^{(2)}} [D^+ H(\| x(s) \|)]_+\, ds.$$

In view of (4), there exists a $\delta_0 > 0$ such that $t_n^{(2)} - t_n^{(1)} \ge \delta_0$, $n = 1.2\ \ldots$ Moreover for $t \in [t_n^{(1)}, t_n^{(2)}]$, we have $\| x(t) \| \ge \upsilon$. The strict uniform boundedness of (2.1) means that for $\upsilon > 0$, there exists a $\beta_2 > 0$ such that $\| x(s) \| \ge \upsilon$ implies that $\| y(t, s, x(s)) \| > \beta_2$, $t \ge s$. Together with (3)

$$0 \le V(t, x(t)) \le V(t_0, y(t, t_0, x_0)) - \int_{t_0}^{t} c(\| y(t, s, x(s)) \|)\, ds$$

$$\le V(t_0, y(t, t_0, x_0)) - \sum_{n=1}^{m} \int_{t_n^{(1)}}^{t_n^{(2)}} c(\| y(t, s, x(s)) \|)\, ds$$

$$\leq \max_{\|y\| \leq \beta_1} V(t_0, y) - c(\beta_2)\, m\, \delta_0 \to -\infty, m \to \infty.$$

This is a contradiction and we prove $\limsup_{t \to \infty} \|x(t, t_0, x_0)\| = 0$.

Thus we complete the proof by combining the two claims.

*Remark* : If $H(y) = y$ and $\|F(t, x)\| \leq M$ where $M$ is positive, then condition (4) in the above theorem holds and thus the former results are the special cases.

Finally let us give two examples in illustration of uniform boundedness and ultimate uniform boundedness of solutions respectively.

*Example* 4.1 — Consider the differential equation

$$x'' + (1 + f(t))\, x = 0.$$

Assume that $f(t)$ is continuous in $R_+$ and $\int_0^\infty |f(\sigma)|\, d\sigma < \infty$. Through transformation, we get the equivalent differential equations

$$\begin{cases} x_1' = x_2, \\ x_2' = -(1 + f(t))\, x_1. \end{cases} \qquad \text{... (4.1)}$$

Consider the corresponding nonperturbed differential equations

$$\begin{cases} x_1' = x_2, \\ x_2' = -x_1, \end{cases} \qquad \text{... (4.2)}$$

which has a solution $y(t, t_0, x_0) = (y_1(t, t_0, x_0), y_2(t, t_0, x_0))$ with the initial value $x_0 = (x_{01}, x_{02})$, where

$$\begin{cases} y_1(t, t_0, x_0) = x_{01} \cos(t - t_0) + x_{02} \sin(t - t_0), \\ y_2(t, t_0, x_0) = -x_{01} \sin(t - t_0) + x_{02} \cos(t - t_0). \end{cases}$$

Obviously the system (4.2) is strictly uniformly bounded.

Let $\quad V(t, x_1, x_2) = e^{-\int_0^t |f(\sigma)|\, d\sigma}(x_1^2 + x_2^2)$. Obviously,

$$V(t, x_1, x_2) \leq x_1^2 + x_2^2 = a(\|x\|),$$

$$V(t, x_1, x_2) \geq e^{-\int_0^\infty |f(\sigma)|\, d\sigma}(x_1^2 + x_2^2) = b(\|x\|).$$

Thus $V$ satisfies (1) (2) of Theorem 4.1.

$$\frac{dV}{ds} = (y_1^2(t, s, x(s)) + y_2^2(t, s, x(s)))\, e^{-\int_0^s |f(\sigma)|\, d\sigma} (-|f(s)|)$$

$$+ e^{-\int_0^s |f(\sigma)|\, d\sigma}\, 2(x_1(s), x_2(s)) \begin{pmatrix} 0 \\ -f(s)\, x_1(s) \end{pmatrix}$$

$$= e^{-\int_0^s |f(\sigma)|\, d\sigma} [-|f(s)|(x_1^2(s) + x_2^2(s)) - 2f(s)x_1(s)x_2(s)]$$

$$= -e^{-\int_0^s |f(\sigma)|\, d\sigma} [|f(s)|(x_1^2(s) + x_2^2(s)) + 2f(s)x_1(s)x_2(s)]$$

$$\leq 0.$$

By Theorem 4.1, we deduce that the system (4.1) is uniformly bounded.

*Example* 4.2 — Consider the differential equations

$$\begin{cases} x_1' = x_2 - ax_1(x_2^2 + x_2^2), \\ x_2' = -x_1 - ax_2(x_1^2 + x_2^2), \quad a > 0 \end{cases} \qquad \text{... (4.3)}$$

its corresponding nonperturbed differential equations

$$\begin{cases} x_1' = x_2, \\ x_2' = -x_1, \end{cases} \qquad \text{... (4.4)}$$

which has a solution $y(t, t_0, x_0) = (y_1(t, t_0, x_0), y_2(t, t_0, x_0))$ with the initial value $x_0 = (x_{01}, x_{02})$, where

$$\begin{cases} y_1(t, t_0, x_0) = x_{01} \cos(t - t_0) + x_{02} \sin(t - t_0), \\ y_2(t, t_0, x_0) = -x_{01} \sin(t - t_0) + x_{02} \cos(t - t_0). \end{cases}$$

Obviously the system (4.4) is strict uniform boundedness.

Let $\quad V(x_1, x_2) = x_1^2 + x_2^2.$

Clearly $V$ satisfies (1) (2) of Theorem 4.2.

$$\frac{dV}{ds} = 2\,(y_1(t, s, x(s)), y_2(t, s, x(s))) \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}$$

$$\begin{pmatrix} -ax_1(s)(x_1^2(s) + x_2^2(s)) \\ -ax_2(s)(x_1^2(s) + x_2^2(s)) \end{pmatrix}$$

$$= -2a(x_1^2(s) + x_2^2(s))^2 = -2a(\| y(t, s, x(s)) \|)^4.$$

By Theorem 4.2, we deduce that the system (4.3) is ultimately uniformly bounded.

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