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FUNCTIONAL CENTRAL LIMIT THEOREMS FOR ITERATED FUNCTION SYSTEMS CONTROLLED BY REGENERATIVE SEQUENCES*

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Let X be a Polish space and let S be a measurable space. Let $\{I_n\}$ be a regenerative process with state space s. Take Z_0 arbitrary but independent of $\{I_n\}$. We consider an iterated function system obtained recursively by $Z_n = F_{I_{n-1}}(Z_{n-1})\ (n \geq 1)$, where the function $F: X \times S \to X$ defined by $F(x, s) = F_s(x)$ is measurable and for each $s \in S$, $F_s(x)$ is a continuous function of x. We obtain sufficient conditions under which, whatever the initial distribution, the functional central limit theorem holds.

Key Words : Regenerative Process; Markov Chain; Invariant Probability; Functional Central Limit Theorem

1. INTRODUCTION AND PRELIMINARIES

Let X be a Polish space with metric d, and S be an arbitrary measurable space. Let $F: X \times S \to X$ be a measurable function. Assume that for each $s \in S$, the function $F_s(x) = F(x, s)$ is continuous with respect to x. Let $\left\{I_n\right\}_{n=0}^{\infty}$ be a stochastic sequence with state space S. For specified initial value $x \in X$, we define

$$Z_0(x) = x$$

and

$$Z_n(x) := F(Z_{n-1}(x), I_{n-1}) = F_{I_{n-1}}(Z_{n-1}(x)),\ (n \geq 1). \quad \text{... (1.1)}$$

This particular type of stochastic dynamical system $\left\{Z_n(x)\right\}_{n=0}^{\infty}$ is called an IFS (iterated function system) controlled by $\{I_n\}$. Systems of this type have been studied in many papers (see, e.g. [2]-[4], [7], [10-12], [14], [15] and the references therein). For the case that $\{I_n\}$ is a sequence of i.i.d. random variables, ergodicitiy, law of large numbers, (functional) central limit theorem, and other limiting properties are investigated in, for example, [3], [4], [10], [11]. In [2], [7], and [14],

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iterated function systems given by more general controlling sequences, such as, stationary sequence, semi-Markov chain or regenerative sequence are considered.

Suppose that $\left\{X_n\right\}_{n=0}^{\infty}$ is a Markov chain with state space X, transition probability function $p(x, dy)$, and the unique invariant initial distribution μ. If the distribution of X_0 is the invariant distribution μ, then $\left\{X_n\right\}_{n=0}^{\infty}$ is a stationary ergodic Markov chain. Let P be the transition operator on $L^2(X, \mu)$ defined by

$$Pf(x) := \int f(y)p(x, dy), \quad f \in L^2(X, \mu).$$

Let $P^n f(x) := P(P^{n-1} f)(x)$ and let I denote the identity operator. We will denote the L^2-norm with respect to the invariant distribution by $\|\cdot\|_2$. Write $\bar{f} = \int f \, d\mu$.

If $\quad (P - I)\, g = f - \bar{f}$, then

$$\sum_{j=0}^{n} (f(X_j) - \bar{f}) = \sum_{j=0}^{n} (Pg(X_j) - g(X_j))$$

$$= \sum_{j=1}^{n+1} (Pg(X_{j-1}) - g(X_j)) + (g(X_{n+1}) - g(X_0)).$$

Since $Pg(X_{j-1}) - g(X_j), j \geq 1$ is a stationary ergodic sequence of martingale differences, the FCLT (functional central limit theorem) holds for $f \in L^2(X, \mu)$, i.e. the sequence of stochastic processes $Y_n(t)$ given by

$$Y_n(t) = n^{-\frac{1}{2}} \sum_{j=0}^{[nt]} (f(X_j) - \bar{f}), \quad (t \geq 0)$$

converges in distribution to a Brownian motion with mean zero and variance parameter $\| g \|_2^2 - \| Pg \|_2^2$ (see, e.g. [6], [9], [11]). Here $[nt]$ is the integer part of nt.

Proposition 1.1 — Let $\{X_n\}$ be a sequence of stationary ergodic Markov chain with unique invariant probability μ. If for $f \in L^2(X, \mu), f - \bar{f}$ is in the range of $P - I$, then the *FCLT* holds for f.

PROOF : A proof is given by Gordin and Lipšic[9].

The next lemma is simple but crucial for proving theorems in section 2.

Lemma 1.1 — Let $f \in L^2(X, \mu)$. If $\sum_{n=0}^{\infty} \| P^n(f - \bar{f}) \|_2 < \infty$, then $f - \bar{f}$ belongs to the range of $P - I$; indeed $(P - I)\, g = f - \bar{f}$ where

$$g = -\sum_{n=0}^{\infty} P^n (f - \bar{f}).$$

PROOF : Apply P to both sides of (1.4).

Our main objective is to obtain the FCLT of IFS $\{Z_n(x)\}$ of (1.1) when controlling sequence $\{I_n\}$ is a regenerative process.

2. MAIN RESULTS

Let X be a polish space with metric d, and S be an arbitrary measurable space. Let $F : X \times S \to X$ be a measurable function and let for each $s \in S$, the function $F_s(x) = F(x, s)$ be continuous with respect to x.

We consider an IFS $\left\{ Z_n(x) \right\}_{n=0}^{\infty}$ given by for initial value x in X,

$$Z_0(x) = x, \; Z_n(x) = F_{I_{n-1}} \circ F_{I_{n-2}} \circ \ldots \circ F_{I_0}(x), \quad (n \geq 1) \qquad \ldots (2.1)$$

where $\{I_n\}$ is a discrete time regenerative process with state space S.

Assume that $0 = T_0 < T_1 < \ldots < T_n < \ldots$ denotes its regeneration moments with $E(T_1) < \infty$.

Let $$J_n := (I_{T_n}, \ldots, I_{T_{n+1}} - 1, T_{n+1} - T_n), (n \geq 0)$$

and $$F_{J_n} := F_{I_{T_{n+1}}} - 1 \circ \ldots \circ F_{I_{T_n}}. \qquad \ldots (2.2)$$

Then $$ZT_n(x) = F_{J_{n-1}} \circ \ldots \circ F_{J_0}(x), (n \geq 1), \; Z_{T_0}(x) = x. \qquad \ldots (2.3)$$

Define $$\tilde{Z}_{T_n}(x) := F_{J_0} \circ F_{J_1} \circ \ldots \circ F_{J_{n-1}}(x), (n \geq 1), \; \tilde{Z}_{T_0}(x) = x. \qquad \ldots (2.4)$$

Since $\left\{ I_n \right\}_{n=0}^{\infty}$ is a regenerative sequence, $\left\{ J_n \right\}_{n=0}^{\infty}$ is an i.i.d. sequence and hence $\{Z_{T_n}(x)\}$ is a homogeneous Markov chain. Moreover, $Z_{T_n}(x)$ and $\tilde{Z}_{T_n}(x)$ have the same distribution.

Let (X, d) and (Y, d^*) be two metric spaces. Let us, for a function $h : X \to Y$, define a generalized norm

$$\| h \| := \sup_{x, y \in X, x \neq y} \frac{d^*(h(x), h(y))}{d(x, y)} \qquad \ldots (2.5)$$

satisfying

$$d^*(h(x), h(y)) \leq \| h \| \, d(x, y), \quad \forall \, x, y \in X.$$

A function h is called a Lipschitzian function if $\| h \| < \infty$. We shall write R for the set of real numbers.

We make the following assumptions :

(A) $E \ln \| Z_{T_1} \| < 0.$

(B) $E \ln^+ d(x, Z_{T_1}(x)) < \infty$ for some $x \in \boldsymbol{X}$.

(C) $E \| Z_{T_1} \|^2 := \lambda < 1.$

(D) $Ed\,(x_0, Z_{T_1}(x_0))^2 := d < \infty$ for some $x_0 \in \boldsymbol{X}$.

Note that if (C) and (D) hold, so do (A) and (B). We use the time reversal technique to prove the following two lemmas.

Lemma 2.1 — Under the conditions (A) and (B), the Markov chain $\{Z_{T_n}(x)\}$ has a unique stationary probability measure μ, and the distribution of $Z_{T_n}(x)$ converges weakly to μ as $n \to \infty$, for all $x \in \boldsymbol{X}$.

PROOF : $\{Z_{T_n}(x)\}$ can be considered as an IFS obtained by successive compositions of a sequence of ii.d. random variables and proofs can be found in [7] and [14].

Lemma 2.2 — Suppose the assumptions (C) and (D) hold. Then

$$\forall\ x \in \boldsymbol{X},\ \int d(x, y)^2\, \mu\,(dy) < \infty, \qquad \text{... (2.6)}$$

and

$$\int \int d(x, y)^2\, \mu\,(dx)\, \mu\,(dy) < \infty. \qquad \text{... (2.7)}$$

PROOF : Since $d(x_0, y)$ is a continuous function of y and the distribution of $Z_{T_n}(x_0)$ converges weakly to μ, we have

$$\int d(x_0, y)^2\, \mu\,(dy) \leq \liminf_n\ Ed(Z_{T_n}(x_0), x_0)^2.$$

Let $M_n := \left(Ed\,(Z_{T_n}(x_0), x_0)^2 \right)^{\frac{1}{2}}$. Using that $\{J_n\}$ is i.i.d. and applying successively the triangle inequality and the Minkowski's inequality, we get

$$\begin{aligned} M_n &= \left(Ed\,(\tilde{Z}_{T_n}(x_0), x_0)^2 \right)^{\frac{1}{2}} \\ &\leq \left(Ed\,(\tilde{Z}_{T_n}(x_0), \tilde{Z}_{T_{n-1}}(x_0))^2 \right)^{\frac{1}{2}} + \left(Ed\,(\tilde{Z}_{T_{n-1}}(x_0), x_0)^2 \right)^{\frac{1}{2}} \\ &\leq \prod_{i=0}^{n-2} \left(E \| \boldsymbol{F}_{J_i} \|^2 \right)^{\frac{1}{2}} \left((Ed\, \boldsymbol{F}_{J_{n-1}}(x_0), x_0)^2 \right)^{\frac{1}{2}} + M_{n-1} \end{aligned}$$

$$= \lambda^{\frac{n-1}{2}} d^{\frac{1}{2}} + M_{n-1} \qquad \text{... (2.8)}$$

$$\leq \lambda^{\frac{n-1}{2}} \cdot d^{\frac{1}{2}} + \lambda^{\frac{n-2}{2}} \cdot d^{\frac{1}{2}} + \ldots + \lambda^{\frac{1}{2}} d^{\frac{1}{2}} + d^{\frac{1}{2}}$$

$$\leq d^{\frac{1}{2}} \left(\frac{1}{1-\sqrt{\lambda}} \right) < \infty, \qquad \text{... (2.9)}$$

for all $n, n \geq 1$.

From (2.8) and (2.9),

$$\int d(x_0, y)^2 \mu (dy) < \infty. \qquad \text{... (2.10)}$$

Next, we have, by (2.9) and (2.10)

$$\left(\int d(x, y)^2 \mu (dy) \right)^{\frac{1}{2}} \leq \liminf_n \left(Ed(Z_{T_n}(x_0), x^2 \right)^{\frac{1}{2}}$$

$$\leq \liminf_n M_n + d(x, x_0)$$

$$\leq d^{\frac{1}{2}} \left(\frac{1}{1-\sqrt{\lambda}} \right) + d(x, x_0) < \infty, \qquad \text{... (2.11)}$$

for each x in X and from that (2.6) follows.

Combining (2.10) and (2.11), we obtain (2.7).

We set $\qquad Z_n^{\mu} := Z_n(Z_0), (n \geq 1), Z_0^{\mu} = Z_0$

for X-valued random variable Z_0 with invariant initial distributon μ independent of $\left\{ I_n \right\}_{n=0}^{\infty}$

Theorem 2.1 — *Assume (C) and (D) hold. Then* $\left\{ Z_{T_n}^{\mu} \right\}_{n=0}^{\infty}$ *is a stationary ergodic Markov chain and hence for every Lipschitzian function* $f : X \to R$, *the FCLT holds for*

$$Y_n(t) = n^{-\frac{1}{2}} \sum_{j=0}^{[nt]} \left(f(Z_{T_j}^{\mu}) - \bar{f} \right) \quad (t \geq 0). \qquad \text{... (2.12)}$$

PROOF : By Lemma 2.1, $\left\{ Z_{T_n}^{\mu} \right\}$ with invariant initial distribution μ is a stationary ergodic Markov chain. For Lipschitzian function $f : X \to R$, $|f(y)| \leq \|f\| d(x_0, y) + |f(x_0)|$ for $x_0 \in X$. This together with Lemma 2.2, induces that $f \in L^2(X, \mu)$.

Now

$$|P^n(f-\bar{f})(x)|^2 \leq \int |E[f(Z_{T_n}(x))] - E[f(Z_{T_n}(y))]|^2 \mu(dy)$$

$$\leq \int E|f(Z_{T_n}(x)) - f(Z_{T_n}(y))|^2 \mu(dy)$$

$$\leq \|f\|^2 \int Ed(Z_{T_n}(x), Z_{T_n}(y))^2 \mu(dy). \quad \ldots (2.13)$$

But
$$Ed(Z_{T_n}(x), Z_{T_n}(y))^2 \leq \prod_{i=0}^{n-1} E\|F_{J_i}\|^2 d(x,y)^2 = \lambda^n d(x,y)^2. \quad \ldots (2.14)$$

Combining (2.13) and (2.14) we obtain that

$$\|P^n(f-\bar{f})\|_2 = \left(\int |P^n(f-\bar{f})(x)|^2 \mu(dx)\right)^{\frac{1}{2}}$$

$$\leq \|f\| \left(\int\int Ed(Z_{T_n}(x), Z_{T_n}(y)) = \mu(dy)\,\mu(dx)\right)^{\frac{1}{2}}$$

$$\leq \|f\| \lambda^{\frac{n}{2}} \left(\int\int d(x,y)^2 \mu(dy)\,\mu(dx)\right)^{\frac{1}{2}}.$$

This implies that

$$\sum_{n=0}^{\infty} \|P^n(f-\bar{f})\|_2 \leq \frac{\|f\|M}{1-\sqrt{\lambda}} < \infty, \quad \ldots (2.15)$$

where $M := \left(\int\int d(x,y)^2 \mu(dy)\,\mu(dx)\right)^{\frac{1}{2}} < \infty$, by Lemma 2.2. Hence, by Proposition 1.1 and Lemma 1.1, the FCLT for $Y_n(t)$ of (2.12) follows.

Assume that

(E) $f: \boldsymbol{X} \to \boldsymbol{R}$ is a Lipschitzian function such that for some $y_0 \in \boldsymbol{X}$,

$$E\left[\sum_{k=0}^{T_1-1} f(Z_k(y_0))\right]^2 := K < \infty. \quad \ldots (2.16)$$

We set
$$\Delta_n := \max_{T_n \leq j < T_{n+1}} \|F_{I_j} \circ \ldots \circ F_{I_{T_n}}\|.$$

Then
$\left\{(T_{n+1}-T_n)\,\Delta_n\right\}_{n=0}^{\infty}$ is a sequence of i.i.d. random variables.

***Theorem* 2.2** — *In addition to the assumptions (C) and (D), we assume that* $E(T_1 \Delta_0)^2 < \infty$. *Then for f which satisfies the condition (E),*

(a) the FCLT holds for

$$Y_n^{\mu}(t) = n^{-\frac{1}{2}} \sum_{j=0}^{[nt]} \left[\sum_{k=T_j-1}^{T_j-1} f(Z_k^{\mu}) - \hat{f} \right], \qquad \dots (2.17)$$

and

(b) for any $x \in X$, the FCLT holds for

$$Y_n^{x}(t) = n^{-\frac{1}{2}} \sum_{j=0}^{[nt]} \left[\sum_{k=T_j-1}^{T_j-1} f Z_k^{(x)} - \hat{f} \right]. \qquad \dots (2.18)$$

Here $$\hat{f} := E\left[\sum_{k=0}^{T_1-1} f(Z_k^{\mu}) \right].$$

To prove the Theorem 2.2, we first consider the followings :

For a function $f: X \times S \to \boldsymbol{R}$, we define

$$\alpha_n := \sum_{k=T_{n-1}}^{T_n-1} f(Z_k^{\mu}, I_k) \quad , n = 1, 2, 3, \dots$$

Then $\left\{ Z_{T_n}^{\mu}, \alpha_n \right\}_{n=1}^{\infty}$ is a Markov chain with transition probability

$$P_x(B, D) = P\left(Z_{T_{n+1}}^{\mu} \in B, \alpha_{n+1} \in D \mid Z_{T_n}^{\mu} = x, \alpha_n = s \right)$$

which does not depend on the second component s, and $\left\{ Z_{T_n}^{\mu}, \alpha_n \right\}_{n=1}^{\infty}$ has the unique invariant probability measure given by

$$\pi(B, D) = \int P_y(B, D)\, \mu(dy). \qquad \dots (2.19)$$

Since the distribution of Z_0 is the invariant initial probability μ, $\left\{ Z_{T_n}^{\mu}, \alpha_n \right\}_{n=1}^{\infty}$ is a stationary ergodic Markov chain with state space $\boldsymbol{X} \times \boldsymbol{R}$.

PROOF OF THEOREM 2.2 : (*a*) Consider the stationary ergodic Markov chain $\left\{ Z_{T_n}^{\mu}, \alpha_n \right\}_{n=1}^{\infty}$ with invariant initial distribution π defined in (2.19).

Let $h: X \times R \to R$ be such that $h(x, r) = r$ and let $f: X \times S \to R$ be such that $f(x, s) = f(x)$ where $f: X \to R$ satisfies the condition (E).

Then followings are obtained :

$$\| h \|_2^2 = \int (h(x, r))^2 \pi(dx, dr)$$

$$= \int E\left[(h(Z^{\mu}_{T_1}, \alpha_1))^2 \mid Z_{T_0} = x \right] \mu(dx)$$

$$= \int E\left[\sum_{k=0}^{T_1 - 1} f(Z_k(x)) \right]^2 \mu(dx) \qquad \text{... (2.20)}$$

for each n, $n \geq 1$ and

$$\sum_{k=T_{n-1}}^{T_n - 1} d(Z_k(x), Z_k(y))$$

$$= d\left(Z_{T_{n-1}}(x), Z_{T_{n-1}}(y) \right) + d\left(F_{I_{T_{n-1}}} \circ Z_{T_{n-1}}(x), F_{I_{T_{n-1}}} \circ Z_{T_{n-1}}(y) \right)$$

$$+ \dots +$$

$$+ d\left(F_{I_{T_n - 2}} \circ \dots \circ F_{I_{T_{n-1}}}(x), F_{I_{T_n - 2}} \circ \dots \circ F_{I_{T_{n-1}}} \circ Z_{T_{n-1}}(y) \right)$$

$$\leq \prod_{i=0}^{n-2} \| F_{J_i} \| d(x, y) [(T_n - T_{n-1}) \Delta_{n-1} + 1]. \qquad \text{... (2.21)}$$

Moreover, using inequality $(a+b)^2 \leq 2a^2 + 2b^2$ $(a, b \in R)$, we have

$$\left[\sum_{k=0}^{T_1 - 1} f(Z_k(x)) \right]^2 \leq 2\left[\| f \| \sum_{k=0}^{T_1 - 1} d(Z_k(x), Z_k(y_0)) \right]^2 + 2\left[\sum_{k=0}^{T_1 - 1} f(Z_k(y_0)) \right]^2 \qquad \text{.... (2.22)}$$

From (2.6), (2.16) and (2.20)-(2.22), we get

$$\| h \|_2^2 = \int E\left[\sum_{k=0}^{T_1 - 1} f(Z_k(x)) \right]^2 \mu(dx)$$

$$\leq 2 \| f \|^2 \int E\left[\sum_{k=0}^{T_1 - 1} d(Z_k(x), Z_k(y_0)) \right]^2 \mu(dx) + 2E\left[\sum_{k=0}^{T_1 - 1} f(Z_k(y_0)) \right]^2$$

$$\le 2\,\|f\|^2 \int E\,[d(x, y_0)\,(T_1\,\Delta_0 + 1)]^2\,\mu\,(dx) + 2K$$

$$= 2\,\|f\|^2\,E\,(T_1\,\Delta_0 + 1)^2 \int d(x, y_0)^2\,\mu\,(dx) + 2K < \infty,$$

and hence $h \in L^2\,(\boldsymbol{X} \times \boldsymbol{R}, \pi)$.

Now for the transition operator P defined on $L^2\,(\boldsymbol{X} \times \boldsymbol{R}, \pi)$, it follows that for any $n \ge 1$,

$$P^n\,h(x, r) = E\left[\sum_{k=T_{n-1}}^{T_n - 1} f(Z_k^{\mu}) \mid Z_0 = x\right] = E\left[\sum_{k=T_{n-1}}^{T_n - 1} f(Z_k\,(x))\right]. \quad \text{... (2.23)}$$

On the other hand, since $Z^{\mu}_{T_n}, n \ge 0$ is a stationary ergodic Markov chain, for any $n \ge 1$,

$$\bar{h} = \int h(x, r)\,\pi\,(dx, dr) = \int E\left[\sum_{k=T_{n-1}}^{T_n - 1} f(Z_k\,(x))\right]\mu\,(dx). \quad \text{... (2.24)}$$

Using eq. (2.21), (2.23) and (2.24), we have

$$|\,P^n\,(h - \bar{h})\,(x, r)\,|^2$$

$$= \left|\int \left(E\left[\sum_{k=T_{n-1}}^{T_n - 1} f(Z_k\,(x))\right] - E\left[\sum_{T_{n-1}}^{T_n - 1} f(Z_k(y))\right]\right)\mu\,(dy)\right|^2$$

$$\le \int E\left[\sum_{k=T_{n-1}}^{T_n - 1} f(Z_k\,(x)) - f(Z_k\,(y))\right]^2 \mu\,(dy)$$

$$\le \int E\left[\prod_{i=0}^{n-2} \|\boldsymbol{F}_{J_i}\|\,d(x, y)\,((T_n - T_{n-1})\,\Delta_{n-1} + 1)\right]^2 \mu\,(dy)$$

$$\le \lambda^{n-1}\,E\Big[\,(T_n - T_{n-1})\,\Delta_{n-1} + 1\,\Big]^2 \int d(x, y)^2\,\mu\,(dy). \quad \text{... (2.25)}$$

From (2.8), (2.23) and the fact that $\{(T_n - T_{n-1})\,\Delta_{n-1}\}$ is i.i.d., we obtain

$$\sum_{n=1}^{\infty} \|\,P^n\,(h - \bar{h})\,\|_2$$

$$\leq \sum_{n=1}^{\infty} \left(\int | P^n (h - \bar{h}) (x, r) |^2 d\pi(x, r) \right)^{\frac{1}{2}}$$

$$= \sum_{n=1}^{\infty} \lambda^{\frac{n-1}{2}} \left(E [(T_n - T_{n-1}) \Delta_{n-1} + 1]^2 \right)^{\frac{1}{2}} \left(\int \int d(x, y)^2 \mu(dx) \mu(dy) \right)^{\frac{1}{2}}$$

$$\leq \left(E (T_1 \Delta_0 + 1)^2 \right)^{\frac{1}{2}} M \frac{1}{1 - \sqrt{\lambda}} < \infty,$$

where M is given in the proof of Theorem 2.1. This implies that by Proposition 1.1 and Lemma 1.1, the FCLT holds for

$$Y_n^{\mu}(t) = n^{-\frac{1}{2}} \sum_{j=0}^{[nt]} (h(Z_{T_j}^{\mu}, \alpha_j) - \bar{h}) = n^{-\frac{1}{2}} \sum_{j=0}^{[nt]} \left(\sum_{k=T_j-1}^{T_j-1} f(Z_k^{\mu}) - \hat{f} \right),$$

where $$\hat{f} = E \left[\sum_{k=0}^{T_1-1} f(Z_k^{\mu}) \right].$$

(b) Note that Z_0 and $\{I_n\}$ are independent. For each x and X

$$E\left(\max_{0 \leq t \leq 1} | Y_n^{\mu}(t) - Y_n^{x}(t) | \right)$$

$$\leq n^{-\frac{1}{2}} \sum_{j=0}^{n} \left[\int E \left(\sum_{k=T_{j-1}}^{T_j-1} | f(Z_k(x)) - f(Z_k(y)) | \right) \mu(dy) \right]$$

$$\leq \| f \| n^{-\frac{1}{2}} \sum_{j=0}^{n} \left[\int E \left(\sum_{k=T_{j-1}}^{T_j-1} d(Z_k(x), Z_k(y)) \right) \mu(dy) \right]$$

$$\leq \| f \| E(T_1 \Delta_0 + 1) \left(\int d(x, y) \mu(dy) \right) \left(n^{-\frac{1}{2}} \sum_{j=0}^{n} \lambda^{\frac{j-1}{2}} \right)$$

$$\to 0 \text{ as } n \to \infty. \qquad \dots (2.26)$$

The third inequality in (2.26) follows from eq. (2.21). Therefore, from (2.26) and the conclusion of part (*a*) of Theorem 2.2, the FCLT for $Y_n^x(t)$ of (2.18) is obtained.

Remark : If $\{I_n\}$ is a sequence of Harris recurrent Markov chain with a general state space S, then $\{Z_n(x)\}$ can be considered as in IFS controlled by the regenerative sequence $\{\tilde{I}_n\}$ which is an artificial Markov chain obtained by the method developed by Athreya and Ney[1]. In this case it is natural to express the conditions related to the regeneration moment T_1 in terms of the transition probabilities of the Markov chain $\{Z_n(x), \tilde{I}_n\}$. This however is not an obvious task, and can constitute a good topic for a further research.

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