

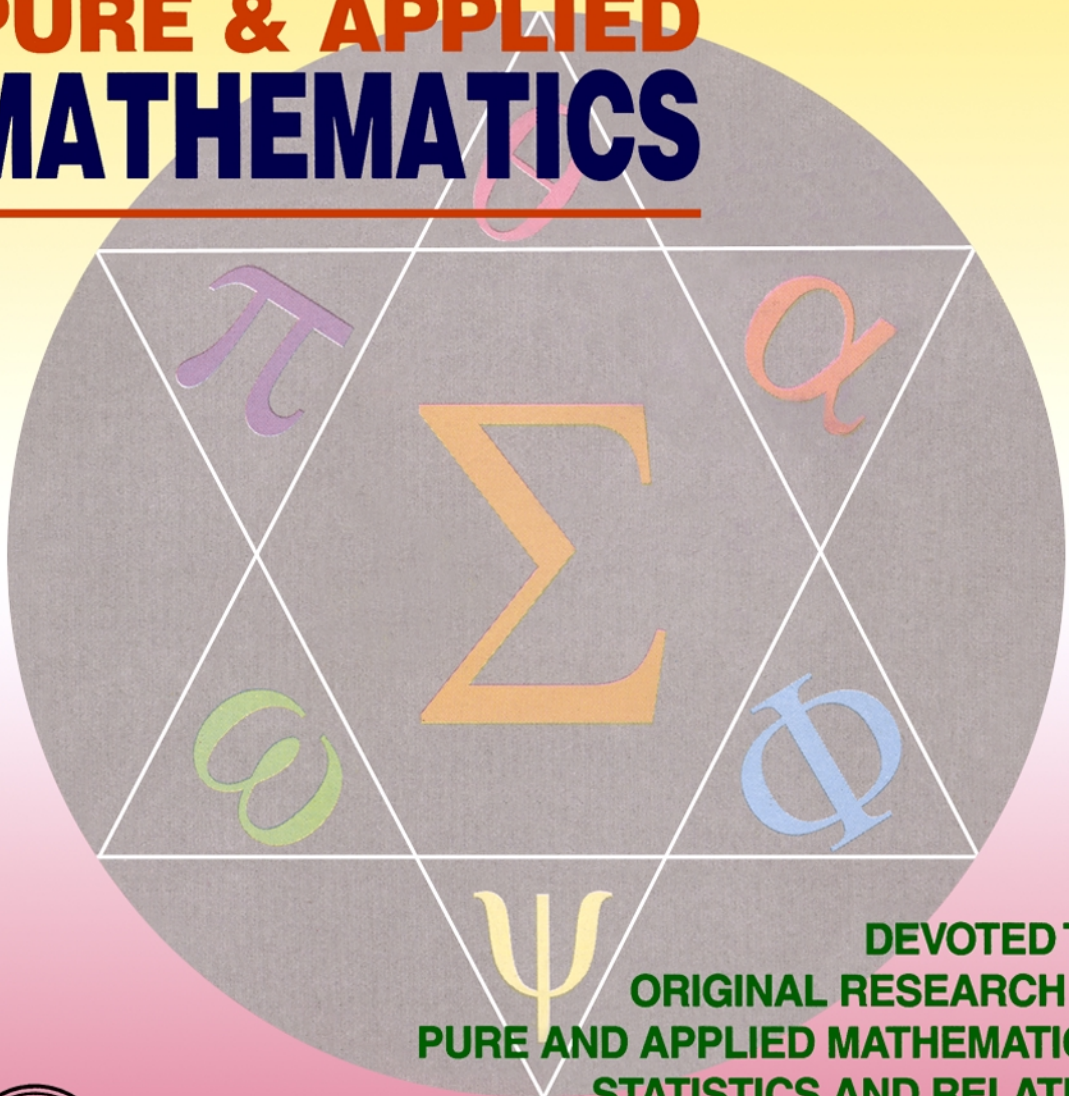
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BASIC HYPERGEOMETRIC SERIES AND CONTINUED FRACTIONS

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In this paper, making use of certain known results, an attempt has been made to express the sum of k th power of the terms of a series as the k th power of a continued fractions.

1. INTRODUCTION

Recently, Singh and Singh³ established the following result :

$$\text{If } A_n = \frac{(abc' d' q^{-1}, ab, ac', ad'; q)_n (1 - abc' d' q^{2n-1})}{(q; q)_n (abc' d'; q)_{2n} (1 - abc' d' q^{-1}) a^n} \\ \times {}_4\Phi_3 \left[\begin{matrix} q^{-n}, abc' d' q^{n-1}, ac, ad; q; q \\ ac', ad', abcd \end{matrix} \right] \quad \dots (1)$$

then for $|\lambda_i/b| < 1$, we have

$$\prod_{i=1}^k {}_3\Phi_2 \left[\begin{matrix} ab, bc, bd; q; \lambda_i/b \\ b \lambda_i, abcd \end{matrix} \right] \\ = \sum_{n=0}^{\infty} A_n^k (-1)^{kn} q^{k \binom{n}{2}} \\ \prod_{i=1}^k \left\{ \frac{(\lambda_i)^n}{(b \lambda_i; q)_n} {}_3\Phi_2 \left[\begin{matrix} abq^n, bc', q^n, bd', q^n; q; \lambda_i/b \\ b \lambda_i q^n, abc' d' q^{2n} \end{matrix} \right] \right\}. \quad \dots (2)$$

In this paper, making use of (1.2) and a known result due to Singh², an attempt has made to express the sum of k th power of the terms of a series as the k th power of a continued fraction. We shall make use of the following known result due to Singh² in order to establish our main result.

$$\begin{aligned}
& \frac{{}_3\Phi_2 \left[\begin{matrix} a, b, c; \frac{de}{abc} \\ d, e \end{matrix} \right]}{{}_3\Phi_2 \left[\begin{matrix} aq, b, c; \frac{de}{abc} \\ dq, e \end{matrix} \right]} = 1 - \frac{\frac{de}{abc} (a-d) (1-b) (1-c) / (1-e) (1-d) (1-dq)}{\frac{(1-e/aq)}{(1-e)} +} \\
& \frac{\frac{e}{aq} (1-aq) \left(1 - \frac{dq}{b}\right) \left(1 - \frac{dq}{c}\right) / (1-e) (1-dq) (1-dq^2)}{1 -} \\
& \frac{\frac{deq}{abc} (a-dq) (1-bq) (1-cq) / (1-eq) (1-dq^2) (1-dq^3)}{\frac{(1-e/aq)}{(1-eq)} +} \\
& \frac{\frac{e}{aq} (1-aq^2) \left(1 - \frac{dq^2}{b}\right) \left(1 - \frac{dq^2}{c}\right) / (1-eq) (1-dq^3) (1-dq^4)}{1 -} \\
& \frac{\frac{deq^2}{abc} (a-dq^2) (1-bq^2) (1-cq^2) / (1-eq^2) (1-dq^4) (1-dq^5)}{\frac{1-e/aq}{(1-eq^2)} +} \dots (2)
\end{aligned}$$

2. DEFINITIONS AND NOTATIONS

The basic hypergeometric series is defined as:

$${}_A\Phi_B \left[\begin{matrix} (a); q; z \end{matrix} \right] = \sum_{r=0}^{\infty} \frac{[(a); q]_r z^r}{[(b); q]_r (q; q)_r}, \quad (|z| < 1, |q| < 1), \quad \dots (3)$$

where $(a; q)_r = (1-a) (1-aq) (1-aq^2) \dots (1-aq^{r-1})$,

$$(a; q)_0 = 1$$

and (a) stands for the sequence of A -parameters of the form $a_1, a_2, \dots, a_A$. The other notations appearing in this paper carry their usual meaning.

3. MAIN RESULTS

In this section we shall establish our main result

Taking $\lambda_i = \lambda$ for $i = 1, 2, \dots, k$ in (1.2), we get

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \frac{(abc' d' q^{-1}, ab, ac', ad'; q)_n (1 - abc' d' q^{2n-1})}{(q; q)_n (abc' d'; q)_{2n} (1 - abc' d' q^{-1}) a^n} \right. \\
& {}_4\Phi_3 \left[\begin{matrix} q^{-n}, abc' d' q^{n-1}, ac, ad; q; q \\ ac', ad', abcd \end{matrix} \right] \frac{\lambda^n}{(b\lambda; q)_n} \\
& \times {}_3\Phi_2 \left[\begin{matrix} abq^n, bc' q^n, bd', q^n; q; \frac{\lambda}{b} \\ b\lambda, abc' d' q^{2n} \end{matrix} \right] \Bigg]^k = \left\{ {}_3\Phi_2 \left[\begin{matrix} ab, bc, bd; q; \frac{\lambda}{b} \\ b\lambda, abcd \end{matrix} \right] \right\}^k \quad (4)
\end{aligned}$$

Replacing a by q/b in (3.4), we get

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \left(\frac{\lambda b}{q} \right)^n \frac{(c' d', q, c' q/b, d' q/b; q)_n (1 - c' d' q^{2n})}{(\lambda b, q; q)_n (c' d' q; q)_{2n} (1 - c' d')} \right. \\
& {}_4\Phi_3 \left[\begin{matrix} q^{-n}, c' d' q^n, cq/b, dq/b; q; q \\ c' q/b, d' q/b, cdq \end{matrix} \right] {}_3\Phi_2 \left[\begin{matrix} q^{n+1}, bc' q^n, bd' q^n; q; \frac{\lambda}{b} \\ b\lambda q^n, c' d' q^{2n+1} \end{matrix} \right] \Bigg]^k \\
& = \left\{ {}_3\Phi_2 \left[\begin{matrix} q, bc, bd; q; \frac{\lambda}{b} \\ b\lambda, cdq \end{matrix} \right] \right\}^k
\end{aligned}$$

Putting $a = 1$ in (1.2) we find

$$\begin{aligned}
& {}_3\Phi_2 \left[\begin{matrix} q, b, c; q; de/bc \\ dq, e \end{matrix} \right] \\
& = \frac{1}{1 - \frac{\frac{de}{bc} (1-d)(1-b)(1-c)/(1-e)(1-d)(1-dq)}{\left(\frac{1-e/q}{1-e} \right) +}} \\
& \frac{\frac{e}{q} (1-q)(1-dq/b)(1-dq/c)/(1-e)(1-dq)(1-dq^2)}{1 -} \\
& \frac{\frac{deq}{bc} (1-dq)(1-bq)(1-cq)/(1-eq)(1-dq^2)(1-dq^3)}{\left(\frac{1-e/q}{1-eq} \right) +} \\
& \frac{\frac{e}{q} (1-q^2)(1-dq^2/b)(1-dq^2/c)/(1-eq)(1-dq^3)(1-dq^4)}{1 -}
\end{aligned}$$

$$\frac{\frac{deq^2}{bc} (1-dq^2) (1-bq^2) (1-cq^2)/(1-eq^2) (1-dq^4) (1-dq^5)}{\left(\frac{1-e/q}{1-eq^2}\right)^+} \quad \dots (6)$$

Putting bc for c , bd for b , $b\lambda$ for e , cd for d in (3.6) we get :

$$\begin{aligned} & {}_3\Phi_2 \left[\begin{matrix} q, bc, bd; q; \lambda/b \\ b\lambda, cdq \end{matrix} \right] \\ &= \frac{1}{1 - \frac{\lambda}{b} (1-cd) (1-bd) (1-bc)/(1-b\lambda) (1-cd) (1-cdq)} \\ & \quad \left(\frac{1-b\lambda/q}{1-b\lambda} \right)^+ \\ & \quad \frac{\frac{b\lambda}{q} (1-q) (1-cq/b) (1-dq/b)/(1-b\lambda) (1-cdq) (1-cdq^2)}{1 -} \\ & \quad \frac{\frac{\lambda q}{b} (1-cdq) (1-bdq) (1-bcq)/(1-b\lambda q) (1-cdq^2) (1-cdq^3)}{\left(\frac{1-b\lambda/q}{1-b\lambda q} \right)^+} \\ & \quad \frac{\frac{b\lambda}{q} (1-q^2) (1-cq^2/b) (1-dq^2/b)/(1-b\lambda q) (1-cdq^3) (1-cdq^4)}{1 -} \\ & \quad \frac{\frac{\lambda q^2}{b} (1-cdq^2) (1-bdq^2) (1-bcq^2)/(1-b\lambda q^2) (1-cdq^4) (1-cdq^5)}{\left(\frac{1-b\lambda/q}{1-b\lambda q^2} \right)^+} \quad \dots (7) \end{aligned}$$

Replacing ${}_3\Phi_2$ function on the right hand side of (3.5) by its equivalent continued traction given in (3.7) we get;

$$\begin{aligned} & \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} (\lambda bq^{-1})^n \frac{(c'd', c'q/b, d'q/b; q)_n (1-c'd' q^{2n})}{(b\lambda; q)_n (c'd' q; q)_{2n} (1-c'd')} \right. \\ & \times {}_4\Phi_3 \left[\begin{matrix} q^{-n}, c'd' q^n, cq/b, dq/b; q; q \\ \frac{c'q}{b}, \frac{d'q}{b}, cdq \end{matrix} \right] {}_3\Phi_2 \left[\begin{matrix} q^{n+1}, bc' q^n, bd' q^n; q; \frac{\lambda}{b} \\ b\lambda q^n, c'd' q^{2n+1} \end{matrix} \right] \Bigg]^k \end{aligned}$$

$$\begin{aligned}
&= \left\{ \frac{1}{1 - \frac{\frac{\lambda}{b} (1 - cd) (1 - bd) (1 - bc) / (1 - b \lambda) (1 - cd) (1 - cdq)}{\left(\frac{1 - b \lambda / q}{1 - b \lambda} \right) +}} \right. \\
&\quad \frac{\frac{b \lambda}{q} (1 - q) (1 - cq/b) (1 - dq/b) / (1 - b \lambda) (1 - cdq) (1 - cdq^2)}{1 -} \\
&\quad \frac{\frac{\lambda q}{b} (1 - cdq) (1 - bdq) (1 - bcq) / (1 - b \lambda q) (1 - cdq^2) (1 - cdq^3)}{\left(\frac{1 - b \lambda / q}{1 - b \lambda q} \right) +} \\
&\quad \left. \frac{\frac{b \lambda}{q} (1 - q^2) (1 - cq^2/b) (1 - dq^2/b) / (1 - b \lambda q) (1 - cdq^3) (1 - cdq^4)}{1 -} \dots \right\}^k \dots (8)
\end{aligned}$$

which is the required result.

4. SPECIAL CASES

Taking $c' = c$ and $d' = \lambda$ in (3.8) and then summing the inner ${}_3\Phi_2$ and ${}_2\Phi_1$ series by making use of [1; App. II (II.12)] and [1; App. II (II.8)] respectively

we get :

$$\begin{aligned}
&\left\{ \frac{(\lambda c, \lambda q/b; q)_\infty}{(\lambda cq \cdot \lambda/b; q)_\infty} \right\}^k \\
&\times \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \frac{(cq/b; q)_n (\lambda/d; q)_n (bcq)_n (1 - \lambda cq^{2n})}{(\lambda b; q)_n (\lambda q/b; q)_n (cdq; q)_n (1 - \lambda c)} (\lambda d)^n \right\}^k \\
&= \left\{ \frac{1}{1 - \frac{\frac{\lambda}{b} (1 - cd) (1 - bd) (1 - bc) / (1 - b \lambda) (1 - cd) (1 - cdq)}{\left(\frac{1 - b \lambda / q}{1 - b \lambda} \right) +}} \right. \\
&\quad \frac{\frac{b \lambda}{q} (1 - q) \left(1 - \frac{cq}{b} \right) \left(1 - \frac{dq}{b} \right) / (1 - b \lambda) (1 - cdq) (1 - cdq^2)}{1 -}
\end{aligned}$$

$$\frac{\frac{\lambda q}{b} (1 - cdq) (1 - bdq) (1 - bcq) / (1 - b \lambda q) (1 - cdq^2) (1 - cdq^3)}{\left(\frac{1 - b \lambda / q}{1 - b \lambda q} \right)^+} + \left. \frac{\frac{b \lambda}{q} (1 - q^2) (1 - cq^2/b) (1 - dq^2/b) / (1 - b \lambda q) (1 - cdq^3) (1 - cdq^4)}{1 - \dots} \right]^k \dots \quad (9)$$

For $k = 1$, we have

$$\begin{aligned} & {}_6\Phi_5 \left[\begin{matrix} q, q \sqrt{\lambda c}, -q \sqrt{\lambda c}, bc, \frac{\lambda}{d}, \frac{cq}{b}, q; -\lambda d \\ \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, cdq, \lambda b; q \end{matrix} \right] \\ &= \lim_{\alpha \rightarrow \infty} {}_7\Phi_6 \left[\begin{matrix} q \sqrt{\lambda c}, -q \sqrt{\lambda c}, bc, \frac{\lambda}{d}, \frac{cq}{b}, \alpha, q; q, \frac{\lambda d}{\alpha} \\ \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, cdq, \lambda b, \frac{\lambda cq}{\alpha} \end{matrix} \right] \\ &= \frac{(1 - \lambda/b)}{(1 - \lambda c)} \left\{ \frac{1}{1 - \frac{\lambda}{b} (1 - cd) (1 - bc) (1 - bd) / (1 - \lambda b) (1 - cd) (1 - cdq)} \right. \\ &\quad \left. \frac{\frac{\lambda q}{b} (1 - cdq) (1 - bcq) (1 - bdq) / (1 - \lambda b q) (1 - cdq^2) (1 - cdq^3)}{\left(\frac{1 - b \lambda / q}{1 - b \lambda q} \right)^+} + \frac{\frac{b \lambda}{q} (1 - q) (1 - cq/b) (1 - dq/b) / (1 - b \lambda) (1 - cdq) (1 - cdq^2)}{1 - \dots} \right\} \\ &\quad \left. \frac{\frac{\lambda q}{b} (1 - cdq) (1 - bcq) (1 - bdq) / (1 - \lambda b q) (1 - cdq^2) (1 - cdq^3)}{\left(\frac{1 - b \lambda / q}{1 - b \lambda q} \right)^+} \dots \right] \dots \quad (10) \end{aligned}$$

For $k = 2$, we have

$$\begin{aligned} & {}_{11}\Phi_{10} \left[\begin{matrix} q, q \sqrt{\lambda c}, -q \sqrt{\lambda c}, bc, \frac{\lambda}{d}, \frac{cq}{b}, q \sqrt{\lambda c}, -q \sqrt{\lambda c}, bc, \frac{\lambda}{d}, \frac{cq}{b}, q; \lambda^2 d^2 \\ \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, cdq, \lambda b, \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, cdq, \lambda b; q^2 \end{matrix} \right] \\ &= \frac{(1 - \lambda/b)^2}{(1 - \lambda c)^2} \left\{ \frac{1}{1 - \frac{\lambda}{b} (1 - cd) (1 - bc) (1 - bd) / (1 - \lambda b) (1 - cd) (1 - cdq)} \right. \\ &\quad \left. \frac{\frac{\lambda q}{b} (1 - cdq) (1 - bcq) (1 - bdq) / (1 - \lambda b q) (1 - cdq^2) (1 - cdq^3)}{\left(\frac{1 - b \lambda / q}{1 - b \lambda q} \right)^+} + \dots \right\} \end{aligned}$$

$$\frac{\frac{\lambda b}{q} (1-q) (1-cq/b) (1-dq/b)/(1-\lambda b) (1-cdq) (1-cdq^2)}{1 - \left[\frac{\frac{\lambda q}{b} (1-cdq) (1-bcq) (1-bdq)/(1-\lambda bq) (1-cdq^2) (1-cdq^3)}{\left(\frac{1-b\lambda/q}{1-b\lambda q} \right)^+} \right]^2} \dots (11)$$

As $d \rightarrow 0$ in (4.10), we obtain

$$\begin{aligned} & {}_5\Phi_4 \left[\begin{matrix} q, q\sqrt{\lambda c}, -q\sqrt{\lambda c}, bc, \frac{cq}{b}; q; \lambda^2 \\ \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, \lambda b; q^2 \end{matrix} \right] \\ &= \left(\frac{1-\lambda/b}{1-\lambda c} \right) \left\{ \frac{1}{1 - \left(\frac{1-\lambda b/q}{1-\lambda b} \right)^+} \frac{\frac{\lambda b}{q} (1-q) (1-cq/b)/(1-\lambda b)}{1 - \frac{\frac{\lambda q}{b} (1-bcq)/(1-\lambda bq) \frac{b\lambda}{q} (1-q^2) (1-cq^2/b)/(1-b\lambda q)}{\left(\frac{1-\lambda b/q}{1-\lambda bq} \right)^+} \dots} \right\} \dots (12) \end{aligned}$$

As $b \rightarrow 1$ and $c \rightarrow \lambda$, (4.12) yields :

$$\begin{aligned} & {}_3\Phi_2 \left[\begin{matrix} q, \lambda q, -\lambda q; \lambda^2 \\ \lambda, -\lambda; q^2 \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(1-\lambda^2 q^{2n}) \lambda^{2n} q^{n(n-1)}}{(1-\lambda^2)} \\ &= \left(\frac{1-\lambda}{1-\lambda^2} \right) \left\{ \frac{1}{1 - \left(\frac{1-\lambda/q}{1-\lambda} \right)^+} \frac{\frac{\lambda}{q} (1-q) (1-\lambda q)/(1-\lambda)}{1 - \frac{\lambda q}{\left(\frac{1-\lambda/q}{1-\lambda q} \right)^+} \frac{\frac{\lambda}{q} (1-q^2) (1-\lambda q^2)/(1-\lambda q)}{1 - \dots}} \right\} \dots (13) \end{aligned}$$

For $\lambda = q$, (4.13) yields :

$$\begin{aligned} & \sum_{n=0}^{\infty} (1-q^{2n+2}) q^{n(n+1)} = (1-q) \left\{ \frac{1}{1-0+} \frac{1}{1-} \frac{(1-q^2)}{1-} \frac{q^2}{0+} \frac{1-q^3}{1-} \dots \right\} \\ &= \frac{(1-q)}{1-} \frac{q}{0+} \frac{1-q^2}{1-} \frac{q^2}{0+} \frac{1-q^3}{1-} \frac{q^3}{0+} \frac{1-q^4}{1-} \frac{q^4}{0+} \dots \dots (14) \end{aligned}$$

If we take $b = 1$, $d = q$ in (3.4) and then representing the ${}_3\Phi_2$ series on the right hand side by its equivalent continued fraction from (3.6) we get :

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^{kn} q^{k \binom{n}{2}} \frac{\lambda^{nk}}{(\lambda; q)_n^k} \left\{ \frac{(ac' d' q^{-1}, a, ac', ad'; q)_n}{(q; q)_n (ac' d'; q)_{2n} a^n} \left(\frac{1 - ac' d' q^{2n-1}}{1 - ac' d' q^{-1}} \right) \right. \\ & \quad \left. {}_4\Phi_3 \left[\begin{matrix} q^{-n}, ac' d' q^{n-1}, ac, aq; q; q \end{matrix} \right] {}_3\Phi_2 \left[\begin{matrix} aq^n, c' q^n, d' q^n; q; \lambda \end{matrix} \right] \right\}^k \\ &= \left\{ \frac{1}{1 - \frac{\lambda(1-ac)(1-a)(1-c)/(1-\lambda)(1-ac)(1-acq)}{\left(\frac{1-\lambda/q}{1-\lambda}\right)^+}} \right. \\ & \quad \frac{\frac{\lambda}{q}(1-q)(1-cq)(1-aq)/(1-\lambda)(1-acq)(1-acq^2)}{1 -} \\ & \quad \left. \frac{\lambda q(1-acq)(1-aq)(1-cq)/(1-\lambda q)(1-acq^2)(1-acq^3)}{\left(\frac{1-\lambda/q}{1-\lambda q}\right)^+} \dots \right\}^k \quad \dots (15) \end{aligned}$$

Taking $d' = \lambda$ and $c' = c$ in (4.15) and summing the inner ${}_3\Phi_2$ and ${}_2\Phi_1$ series by [1; App. II (II.12)] and [1. App. II (II.8)] respectively we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \frac{(\lambda q)^n}{(\lambda; q)_n} \frac{(\lambda acq^{-1}, a, ac; q)_n}{(q, \lambda c; q)_n} \left(\frac{1 - \lambda acq^{2n-1}}{1 - \lambda acq^{-1}} \right) \frac{(\lambda/q, c; q)_n}{(a \lambda, acq; q)_n} \right\}^k \\ &= \left(\frac{(a \lambda c; \lambda; q)_{\infty}}{(\lambda c, \lambda a; q)_{\infty}} \right)^k \left\{ \frac{1}{1 - \frac{\lambda(1-ac)(1-a)(1-c)/(1-\lambda)(1-aac)(1-acq)}{\left(\frac{1-\lambda/q}{1-\lambda}\right)^+}} \right. \\ & \quad \frac{\frac{\lambda}{q}(1-q)(1-cq)(1-aq)/(1-\lambda)(1-acq)(1-acq^2)}{1 -} \\ & \quad \left. \frac{\lambda q(1-acq)(1-aq)(1-cq)/(1-\lambda q)(1-acq^2)(1-acq^3)}{\left(\frac{1-\lambda/q}{1-\lambda q}\right)^+} \dots \right\}^k \quad \dots (16) \end{aligned}$$

For $k = 1$, (4.16) yields :

$$\begin{aligned}
 & {}_7\Phi_6 \left[\begin{matrix} \lambda acq^{-1}, q\sqrt{\lambda acq^{-1}}, -q\sqrt{\lambda acq^{-1}}, a, c, ac, \frac{\lambda}{q}; q; -\lambda q \end{matrix} \right] \\
 &= \frac{(\lambda ac, \lambda; q)_\infty}{(\lambda c, \lambda a; q)_\infty} \left\{ \frac{1}{1 - \frac{\lambda(1-ac)(1-a)(1-c)/(1-\lambda)(1-ac)(1-acq)}{\left(\frac{1-\lambda/q}{1-\lambda}\right)^+}} \right. \\
 &\quad \frac{\frac{\lambda}{q}(1-q)(1-cq)(1-aq)/(1-\lambda)(1-acq)(1-acq^2)}{1 -} \\
 &\quad \left. \frac{\lambda q(1-acq)(1-aq)(1-cq)/(1-\lambda q)(1-acq^2)(1-acq^3)}{\left(\frac{1-\lambda/q}{1-\lambda q}\right)^+} \dots \right\} \dots \quad (17)
 \end{aligned}$$

For $\lambda = q$, (4.17) yields :

$$\begin{aligned}
 & \frac{(aq, cq; q)_\infty}{(q, acq; q)_\infty} = \frac{1}{1 - \frac{q(1-ac)(1-a)(1-c)/(1-q)(1-ac)(1-acq)}{0+}} \\
 & \frac{(1-q)(1-aq)(1-cq)/(1-q)(1-acq)(1-acq^2)}{1 -} \\
 & \frac{q^2(1-acq)(1-aq)(1-cq)/(1-q^2)(1-acq^2)(1-acq^3)}{0+} \dots \dots \quad (18)
 \end{aligned}$$

Replacing a and c by q^a and q^c respectively, we get :

$$\begin{aligned}
 & \frac{(1-q^a)(1-q^c)}{(1-q)(1-q^{a+c})} B_q(a, c) \\
 &= 1 - \frac{q(1-q^a)(1-q^c)/(1-q^{1+a+c})(1-q)}{0+} \\
 & \frac{(1-q^{a+1})(1-q^{c+1})/(1-q^{1+a+c})(1-q^{2+a+c})}{1 -} \\
 & \frac{q^2(1-q^{1+a+c})(1-q^{1+a})(1-q^{1+c})/(1-q^2)(1-q^{2+a+c})(1-q^{3+a+c})}{0+} \dots \dots \quad (19)
 \end{aligned}$$

As $q \rightarrow 1$, (4.19) gives :

$$\left(\frac{ac}{a+c} \right) B(a, c) = \frac{\Gamma[a+1] \Gamma[c+1]}{\Gamma[1+a+c]}$$

$$\begin{aligned}
&= 1 - \frac{ac/(1+a+c)}{0+} \frac{(a+1)(c+1)/(1+a+c)}{1-} \frac{(2+a+c)}{0+} \dots \\
&\quad \frac{(1+a+c)(1+a)(1+c)/2}{0+} \frac{(2+a+c)(3+a+c)}{0+} \dots \quad \dots (20)
\end{aligned}$$

Taking $a = x$, $c = 1 - x$ in (4.20) we get :

$$\begin{aligned}
\frac{x(1-x)\pi}{\sin \pi x} &= 1 - \frac{x(1-x)/2}{0+} \frac{(x+1)(2-x)/6}{1-} \frac{(1+x)(2-x)/12}{0+} \dots \\
&= 1 - \frac{x(1-x)}{0+} \frac{(x+1)(2-x)}{3-} \frac{(1+x)(2-x)}{0+} \frac{(2+x)(3-x)}{5-} \dots \quad \dots (21)
\end{aligned}$$

Taking $x = 1/2$ in (4.21) we get :

$$\begin{aligned}
\frac{\pi}{4} &= 1 - \frac{1/4}{0+} \frac{9/4}{3-} \frac{9/4}{0+} \frac{25/4}{5-} \dots \\
&= 1 - \frac{1}{0+} \frac{9}{3-} \frac{9}{0+} \frac{25}{5-} \frac{25}{0+} \frac{49}{7-} \frac{49}{0+} \dots \quad \dots (22)
\end{aligned}$$

If we take $\lambda \rightarrow q/ac$ in (4.16) we get :

$$\begin{aligned}
&1 + \sum_{n=1}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \frac{(q^2/ac)^n}{(q/ac; q)_n} \frac{(a, c, ac, 1/ac; q)_n}{(q/a, q/c, acq; q)_n} (1+q^n) \right\}^k \\
&= \left\{ \frac{(q, q/ac; q)_{\infty}}{(q/a, q/c; q)_{\infty}} \right\}^k \left[\frac{1 - \frac{q}{ac} (1-ac)(1-a)(1-c)/(1-q/ac)(1-ac)(1-acq)}{1 - \left(\frac{1-1/ac}{1-q/ac} \right)_+} \right. \\
&\quad \left. \frac{1}{ac} \frac{(1-q)(1-cq)(1-aq)/(1-q/ac)(1-acq)(1-acq^2)}{1-} \dots \right]^k \quad \dots (23)
\end{aligned}$$

Taking $a, c \rightarrow \infty$ in (4.23), we obtain :

$$1 + \sum_{n=1}^{\infty} \left\{ (1+q^n) q^{n(3n-1)/2} \right\}^k = \left\{ (q; q)_{\infty} \right\}^k \quad \dots (24)$$

For $k = 1$, (4.24) yields Euler's identity :

$$\sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q; q)_{\infty} \quad \dots (25)$$

For $k = 2$, (4.23) yields :

$$1 + \sum_{n=1}^{\infty} (1+q^n)^2 q^{n(3n-1)} = \left\{ (q; q)_{\infty} \right\}^2. \quad \dots (26)$$

By applying Jacobi's triple product identity, (4.26) can be written as:

$$(q^6; q^6)_{\infty} \left\{ (-q^3; q^6)_{\infty}^2 + (-q^2 - q^4; q^6)_{\infty} \right\} = 1 + (q; q)_{\infty}^2. \quad \dots (27)$$

A number of other interesting results can also be scored.

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