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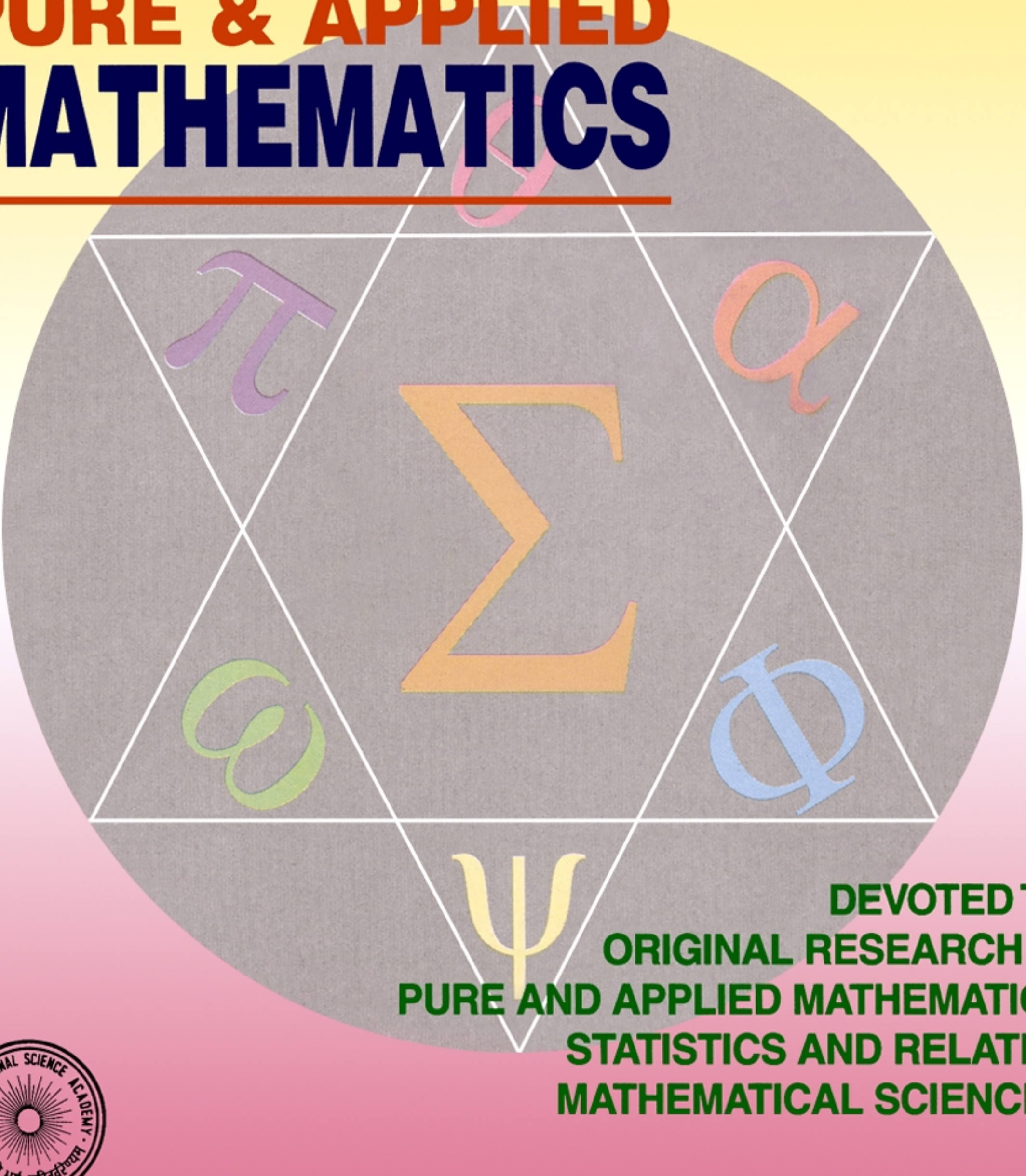

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BASIC HYPERGEOMETRIC SERIES AND CONTINUED FRACTIONS

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In this paper, making use of certain known results, an attempt has been made to express the sum of kth power of the terms of a series as the kth power of a continued fractions.

1. INTRODUCTION

Recently, Singh and Singh[3] established the following result :

$$\text{If } A_n = \frac{(abc'd'q^{-1}, ab, ac', ad'; q)_n (1 - abc'd'q^{2n-1})}{(q; q)_n (abc'd'; q)_{2n} (1 - abc'd'q^{-1}) a^n}$$

$$\times \; {}_4\Phi_3 \left[\begin{matrix} q^{-n}, abc'd'q^{n-1}, ac, ad; q; q \\ ac', ad', abcd \end{matrix} \right] \qquad \dots (1)$$

then for $|\lambda_i/b| < 1$, we have

$$\prod_{i=1}^{k} {}_3\Phi_2 \left[\begin{matrix} ab, bc, bd; q; \lambda_i/b \\ b\lambda_i, abcd \end{matrix} \right]$$

$$= \sum_{n=0}^{\infty} A_n^k (-1)^{kn} q^{k\binom{n}{2}}$$

$$\prod_{i=1}^{k} \left\{ \frac{(\lambda_i)^n}{(b\lambda_i; q)_n} {}_3\Phi_2 \left[\begin{matrix} abq^n, bc'q^n, bd'q^n; q; \lambda_i/b \\ b\lambda_i q^n, abc'd'q^{2n} \end{matrix} \right] \right\}. \qquad \dots (2)$$

In this paper, making use of (1.2) and a known result due to Singh[2], an attempt has made to express the sum of kth power of the terms of a series as the kth power of a continued fraction. We shall make use of the following known result due to Singh[2] in order to establish our main result.

$$\frac{{}_3\Phi_2\left[\begin{matrix} a, b, c; \frac{de}{abc} \\ d, e \end{matrix}\right]}{{}_3\Phi_2\left[\begin{matrix} aq, b, c; \frac{de}{abc} \\ dq, e \end{matrix}\right]} = 1 - \frac{\frac{de}{abc}(a-d)(1-b)(1-c)/(1-e)(1-d)(1-dq)}{\frac{(1-e/aq)}{(1-e)} +}$$

$$\frac{\frac{e}{aq}(1-aq)\left(1-\frac{dq}{b}\right)\left(1-\frac{dq}{c}\right)/(1-e)(1-dq)(1-dq^2)}{1-}$$

$$\frac{\frac{deq}{abc}(a-dq)(1-bq)(1-cq)/(1-eq)(1-sq^2)(1-dq^3)}{\frac{(1-e/aq)}{(1-eq)} +}$$

$$\frac{\frac{e}{aq}(1-aq^2)\left(1-\frac{dq^2}{b}\right)\left(1-\frac{dq^2}{c}\right)/(q-eq)(1-dq^3)(1-dq^4)}{1-}$$

$$\frac{\frac{deq^2}{abc}(a-dq^2)(1-bq^2)(1-cq^2)/(1-eq^2)(1-dq^4)(1-dq^5)}{\frac{1-e/aq)}{(1-eq^2)} +} \qquad \ldots (2)$$

2. Definitions and Notations

The basic hypergeometric series is defined as:

$${}_A\Phi_B\left[\begin{matrix} (a); q; z \\ (b) \end{matrix}\right] = \sum_{r=0}^{\infty} \frac{[(a); q]_r z^r}{[(b); q]_r (q; q)_r}, \; (|z| < 1, |q| < 1), \qquad \ldots (3)$$

where $\quad (a; q)_r = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{r-1}),$

$$(a; q)_0 = 1$$

and (a) stands for the sequence of A-parameters of the form $a_1, a_2, \ldots, a_A$. The other notations appearing in this paper carry their usual meaning.

3. Main Results

In this section we shall establish our main result

Taking $\lambda_i = \lambda$ for $i = 1, 2, \ldots, k$ in (1.2), we get

$$\sum_{n=0}^{\infty}\left\{(-1)^n q^{\binom{n}{2}}\frac{(abc'd'q^{-1}, ab, ac', ad'; q)_n\,(1-abc'd'q^{2n-1})}{(q;q)_n\,(abc'd';q)_{2n}\,(1-abc'd'q^{-1})\,a^n}\right.$$

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, abc'd'q^{n-1}, ac, ad; q; q \\ ac', ad', abcd \end{matrix}\right]\frac{\lambda^n}{(b\lambda;q)_n}$$

$$\left.\times\ {}_3\Phi_2\left[\begin{matrix} abq^n, bc'q^n, bd', q^n; q; \dfrac{\lambda}{b} \\ b\lambda, abc'd'q^{2n} \end{matrix}\right]\right]^k = \left\{{}_3\Phi_2\left[\begin{matrix} ab, bc, bd; q; \dfrac{\lambda}{b} \\ b\lambda, abcd \end{matrix}\right]\right\}^k \tag{4}$$

Replacing a by q/b in (3.4), we get

$$\sum_{n=0}^{\infty}\left\{(-1)^n q^{\binom{n}{2}}\left(\frac{\lambda b}{q}\right)^n \frac{(c'd', q, c'q/b, d'q/b; q)_n\,(1-c'd'q^{2n})}{(\lambda b, q; q)_n\,(c'd'q;q)_{2n}\,(1-c'd')}\right.$$

$$\left. {}_4\Phi_3\left[\begin{matrix} q^{-n}, c'd'q^n, cq/b, dq/b; q; q \\ c'q/b, d'q/b, cdq \end{matrix}\right]{}_3\Phi_2\left[\begin{matrix} q^{n+1}, bc'q^n, bd'q^n; q; \dfrac{\lambda}{b} \\ b\lambda q^n, c'd'q^{2n+1} \end{matrix}\right]\right]^k$$

$$=\left\{{}_3\Phi_2\left[\begin{matrix} q, bc, bd; q; \dfrac{\lambda}{b} \\ b\lambda, cdq \end{matrix}\right]\right\}^k$$

Putting $a = 1$ in (1.2) we find

$$ {}_3\Phi_2\left[\begin{matrix} q, b, c; q; de/bc \\ dq, e \end{matrix}\right]$$

$$=\frac{1}{1-}\,\frac{\frac{de}{bc}(1-d)(1-b)(1-c)/(1-e)(1-d)(1-dq)}{\left(\dfrac{1-e/q}{1-e}\right)+}$$

$$\frac{\frac{e}{q}(1-q)(1-dq/b)(1-dq/c)/(1-e)(1-dq)(1-dq^2)}{1-}$$

$$\frac{\frac{deq}{bc}(1-dq)(1-bq)(1-cq)/(1-eq)(1-dq^2)(1-dq^3)}{\left(\dfrac{1-e/q}{1-eq}\right)+}$$

$$\frac{\frac{e}{q}(1-q^2)(1-dq^2/b)(1-dq^2/c)/(1-eq)(1-dq^3)(1-dq^4)}{1-}$$

$$\cfrac{\frac{deq^2}{bc}(1-dq^2)(1-bq^2)(1-cq^2)/(1-eq^2)(1-dq^4)(1-dq^5)}{\left(\frac{1-e/q}{1-eq^2}\right)+} \qquad \dots (6)$$

Putting bc for c, bd for b, $b\lambda$ for e, cd for d in (3.6) we get :

$${}_3\Phi_2\left[\begin{matrix} q, bc, bd; q; \lambda/b \\ b\lambda, cdq \end{matrix}\right]$$

$$= \frac{1}{1-}\,\cfrac{\frac{\lambda}{b}(1-cd)(1-bd)(1-bc)/(1-b\lambda)(1-cd)(1-cdq)}{\left(\frac{1-b\lambda/q}{1-b\lambda}\right)+}$$

$$\cfrac{\frac{b\lambda}{q}(1-q)(1-cq/b)(1-dq/b)/(1-b\lambda)(1-cdq)(1-cdq^2)}{1-}$$

$$\cfrac{\frac{\lambda q}{b}(1-cdq)(1-bdq)(1-bcq)/(1-b\lambda q)(1-cdq^2)(1-cdq^3)}{\left(\frac{1-b\lambda/q}{1-b\lambda q}\right)+}$$

$$\cfrac{\frac{b\lambda}{q}(1-q^2)(1-cq^2/b)(1-dq^2/b)/(1-b\lambda q)(1-cdq^3)(1-cdq^4)}{1-}$$

$$\cfrac{\frac{\lambda q^2}{b}(1-cdq^2)(1-bdq^2)(1-bcq^2)/(1-b\lambda q^2)(1-cdq^4)(1-cdq^5)}{\left(\frac{1-b\lambda/q}{1-b\lambda q^2}\right)+}. \qquad \dots (7)$$

Replacing ${}_3\Phi_2$ function on the right hand side of (3.5) by its equivalent continued traction given in (3.7) we get;

$$\sum_{n=0}^{\infty}\left\{(-1)^n q^{\binom{n}{2}}(\lambda bq^{-1})^n \frac{(c'd', c'q/b, d'q/b; q)_n (1-c'd'q^{2n})}{(b\lambda; q)_n (c'd'q; q)_{2n} (1-c'd')}\right.$$

$$\left.\times\, {}_4\Phi_3\left[\begin{matrix} q^{-n}, c'd'q^n, cq/b, dq/b; q; q \\ \frac{c'q}{b}, \frac{d'q}{b}, cdq \end{matrix}\right] {}_3\Phi_2\left[\begin{matrix} q^{n+1}, bc'q^n, bd'q^n; q; \frac{\lambda}{b} \\ b\lambda q^n, c'd'q^{2n+1} \end{matrix}\right]\right\}^k$$

$$= \left\{ \frac{1}{1-} \frac{\frac{\lambda}{b}(1-cd)(1-bd)(1-bc)/(1-b\lambda)(1-cd)(1-cdq)}{\left(\frac{1-b\lambda/q}{1-b\lambda}\right)+} \right.$$

$$\frac{\frac{b\lambda}{q}(1-q)(1-cq/b)(1-dq/b)/(1-b\lambda)(1-cdq)(1-cdq^2)}{1-}$$

$$\frac{\frac{\lambda q}{b}(1-cdq)(1-bdq)(1-bcq)/(1-b\lambda q)(1-cdq^2)(1-cdq^3)}{\left(\frac{1-b\lambda/q}{1-b\lambda q}\right)+}$$

$$\left. \frac{\frac{b\lambda}{q}(1-q^2)(1-cq^2/b)(1-dq^2/b)/(1-b\lambda q)(1-cdq^3)(1-cdq^4)}{1-} \cdots \right]^k, \qquad \ldots (8)$$

which is the required result.

4. Special Cases

Taking $c' = c$ and $d' = \lambda$ in (3.8) and then summing the inner ${}_3\Phi_2$ and ${}_2\varphi_1$ series by making use of [1; App. II (II.12)] and [1; App. II (II.8)] respectively
we get :

$$\left\{ \frac{(\lambda c, \lambda q/b; q)_\infty}{(\lambda cq \cdot \lambda/b; q)_\infty} \right\}^k$$

$$\times \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \frac{(cq/b;q)_n (\lambda/d;q)_n (bc\, q)_n}{(\lambda b;.q)_n (\lambda q/b;q)_n (cdq;q)_n} \frac{(1-\lambda cq^{2n})}{(1-\lambda c)} (\lambda d)^n \right\}^k$$

$$= \left\{ \frac{1}{1-} \frac{\frac{\lambda}{b}(1-cd)(1-bd)(1-bc)/(1-b\lambda)(1-cd)(1-cdq)}{\left(\frac{1-b\lambda/q}{1-b\lambda}\right)+} \right.$$

$$\frac{\frac{b\lambda}{q}(1-q)\left(1-\frac{cq}{b}\right)\left(1-\frac{dq}{b}\right)/(1-b\lambda)(1-cdq)(1-cdq^2)}{1-}$$

$$\frac{\frac{\lambda q}{b}(1-cdq)(1-bdq)(1-bcq)/(1-b\lambda q)(1-cdq^2)(1-cdq^3)}{\left(\frac{1-b\lambda/q}{1-b\lambda q}\right)+}$$

$$\left.\frac{\frac{b\lambda}{q}(1-q^2)(1-cq^2/b)(1-dq^2/b)/(1-b\lambda q)(1-cdq^3)(1-cdq^4)}{1-}\cdots\right]^k. \quad \text{... (9)}$$

For $k = 1$, we have

$${}_6\Phi_5\left[\begin{matrix} q, q\sqrt{\lambda c}, -q\sqrt{\lambda c}, bc, \frac{\lambda}{d}, \frac{cq}{b}; q; -\lambda d \\ \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, cdq, \lambda b\,; q \end{matrix}\right]$$

$$= \lim_{\alpha\to\infty} {}_7\Phi_6\left[\begin{matrix} q\sqrt{\lambda c}, -q\sqrt{\lambda c}, bc, \frac{\lambda}{d}, \frac{cq}{b}, \alpha, q; q; \frac{\lambda d}{\alpha} \\ \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, cdq, \lambda b, \frac{\lambda cq}{\alpha} \end{matrix}\right]$$

$$= \frac{(1-\lambda/b)}{(1-\lambda c)}\left\{\frac{1}{1-}\,\frac{\frac{\lambda}{b}(1-cd)(1-bc)(1-bd)/(1-\lambda b)(1-cd)(1-cdq)}{\left(\frac{1-b\lambda/q}{1-b\lambda}\right)+}\right.$$

$$\frac{\frac{b\lambda}{q}(1-q)(1-cq/b)(1-dq/b)/(1-b\lambda)(1-cdq)(1-cdq^2)}{1-}$$

$$\left.\frac{\frac{\lambda q}{b}(1-cdq)(1-bcq)(1-bdq)/(1-\lambda bq)(1-cdq^2)(1-cdq^3)}{\left(\frac{1-b\lambda/q}{1-b\lambda q}\right)+}\cdots\right]. \quad \text{... (10)}$$

For $k = 2$, we have

$${}_{11}\Phi_{10}\left[\begin{matrix} q, q\sqrt{\lambda c}, -q\sqrt{\lambda c}, bc, \frac{\lambda}{d}, \frac{cq}{b}, q\sqrt{\lambda c}, -q\sqrt{\lambda c}, bc, \frac{\lambda}{d}, \frac{cq}{b}; q; \lambda^2 d^2 \\ \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, cdq, \lambda b, \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, cdq, \lambda b; q^2 \end{matrix}\right]$$

$$= \frac{(1-\lambda/b)^2}{(1-\lambda c)^2}\left\{\frac{1}{1-}\,\frac{\frac{\lambda}{b}(1-cd)(1-bc)(1-bd)/(1-\lambda b)(1-cd)(1-cdq)}{\left(\frac{1-b\lambda/q}{1-b\lambda}\right)+}\right.$$

$$\left.\frac{\frac{\lambda b}{q}(1-q)(1-cq/b)(1-dq/b)/(1-\lambda b)(1-cdq)(1-cdq^2)}{1-}\;\frac{\frac{\lambda q}{b}(1-cdq)(1-bcq)(1-bdq)/(1-\lambda bq)(1-cdq^2)(1-cdq^3)}{\left(\frac{1-b\lambda/q}{1-b\lambda q}\right)+}\right]^2 \quad \ldots (11)$$

As $d \to 0$ in (4.10), we obtain

$${}_5\Phi_4\left[\begin{matrix} q, q\sqrt{\lambda c}, -q\sqrt{\lambda c}, bc, \frac{cq}{b}; q; \lambda^2 \\ \sqrt{\lambda c}, -\sqrt{\lambda c}, \frac{\lambda q}{b}, \lambda b; q^2 \end{matrix}\right]$$

$$= \left(\frac{1-\lambda/b}{1-\lambda c}\right)\left\{\frac{1}{1-}\;\frac{\frac{\lambda}{b}(1-bc)(1-\lambda b)}{\left(\frac{1-\lambda b/q}{1-\lambda b}\right)+}\;\frac{\frac{\lambda b}{q}(1-q)(1-cq/b)/(1-\lambda b)}{1-}\right.$$

$$\left.\frac{\frac{\lambda q}{b}(1-bcq)/(1-\lambda bq)}{\left(\frac{1-\lambda b/q}{1-\lambda bq}\right)+}\;\frac{\frac{b\lambda}{q}(1-q^2)(1-cq^2/b)/(1-b\lambda q)}{1-}\cdots\right]. \quad \ldots (12)$$

As $b \to 1$ and $c \to \lambda$, (4.12) yields :

$${}_3\Phi_2\left[\begin{matrix} q, \lambda q, -\lambda q; \lambda^2 \\ \lambda, -\lambda; q^2 \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(1-\lambda^2 q^{2n})\,\lambda^{2n}\,q^{n(n-1)}}{(1-\lambda^2)}$$

$$= \left(\frac{1-\lambda}{1-\lambda^2}\right)\left\{\frac{1}{1-}\;\frac{\lambda}{\left(\frac{1-\lambda/q}{1-\lambda}\right)+}\;\frac{\frac{\lambda}{q}(1-q)(1-\lambda q)/(1-\lambda)}{1-}\right.$$

$$\left.\frac{\lambda q}{\left(\frac{1-\lambda/q}{1-\lambda q}\right)+}\;\frac{\frac{\lambda}{q}(1-q^2)(1-\lambda q^2)/(1-\lambda q)}{1-}\;\cdots\right]. \quad \ldots (13)$$

For $\lambda = q$, (4.13) yields :

$$\sum_{n=0}^{\infty} (1-q^{2n+2})\,q^{n(n+1)} = (1-q)\left\{\frac{1}{1-}\,\frac{1}{0+}\,\frac{(1-q^2)}{1-}\,\frac{q^2}{0+}\,\frac{1-q^3}{1-}\cdots\right\}$$

$$\equiv \frac{(1-q)}{1-}\,\frac{q}{0+}\,\frac{1-q^2}{1-}\,\frac{q^2}{0+}\,\frac{1-q^3}{1-}\,\frac{q^3}{0+}\,\frac{1-q^4}{1-}\,\frac{q^4}{0+}\cdots. \quad \ldots (14)$$

If we take $b=1$, $d=q$ in (3.4) and then representing the ${}_3\Phi_2$ series on the right hand side by its equivalent continued fraction from (3.6) we get :

$$\sum_{n=0}^{\infty} (-1)^{kn} q^{k\binom{n}{2}} \frac{\lambda^{nk}}{(\lambda;q)_n^k} \left\{ \frac{(ac'd'q^{-1}, a, ac', ad'; q)_n}{(q;q)_n (ac'd';q)_{2n} a^n} \left(\frac{1-ac'd'q^{2n-1}}{1-ac'd'q^{-1}} \right) \right.$$

$$\left. {}_4\Phi_3\left[\begin{matrix} q^{-n}, ac'd'q^{n-1}, ac, aq; q; q \\ ac', ad', acq \end{matrix}\right] {}_3\Phi_2\left[\begin{matrix} aq^n, c'q^n, d'q^n; q; \lambda \\ \lambda q^n, ac'd'q^{2n} \end{matrix}\right] \right]^k$$

$$= \left\{ \frac{1}{1-} \frac{\lambda(1-ac)(1-a)(1-c)/(1-\lambda)(1-ac)(1-acq)}{\left(\frac{1-\lambda/q}{1-\lambda}\right)+} \right.$$

$$\frac{\frac{\lambda}{q}(1-q)(1-cq)(1-aq)/(1-\lambda)(1-acq)(1-acq^2)}{1-}$$

$$\left. \frac{\lambda q(1-acq)(1-aq)(1-cq)/(1-\lambda q)(1-acq^2)(1-acq^3)}{\left(\frac{1-\lambda/q}{1-\lambda q}\right)+} \cdots \right]^k. \qquad \ldots (15)$$

Taking $d'=\lambda$ and $c'=c$ in (4.15) and summing the inner ${}_3\Phi_2$ and ${}_2\Phi_1$ series by [1; App. II (II.12)] and [1. App. II (II.8)] respectively we get,

$$\sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \frac{(\lambda q)^n}{(\lambda;q)_n} \frac{(\lambda acq^{-1}, a, ac; q)_n}{(q, \lambda c; q)_n} \left(\frac{1-\lambda acq^{2n-1}}{1-\lambda acq^{-1}} \right) \frac{(\lambda/q, c; q)_n}{(a\lambda, acq; q)_n} \right\}^k$$

$$= \left(\frac{(a\lambda c; \lambda; q)_\infty}{(\lambda c, \lambda a; q)_\infty} \right)^k \left\{ \frac{1}{1-} \frac{\lambda(1-ac)(1-a)(1-c)/(1-\lambda)(1-aac)(1-acq)}{\left(\frac{1-\lambda/q}{1-\lambda}\right)+} \right.$$

$$\frac{\frac{\lambda}{q}(1-q)(1-cq)(1-aq)/(1-\lambda)(1-acq)(1-acq^2)}{1-}$$

$$\left. \frac{\lambda q(1-acq)(1-aq)(1-cq)/(1-\lambda q)(1-acq^2)(1-acq^3)}{\left(\frac{1-\lambda/q}{1-\lambda q}\right)+} \cdots \right]^k \qquad \ldots (16)$$

For $k = 1$, (4.16) yields :

$$ {}_7\Phi_6\left[\begin{matrix} \lambda acq^{-1}, q\sqrt{\lambda acq^{-1}}, -q\sqrt{\lambda acq^{-1}}, a, c, ac, \frac{\lambda}{q}; q; -\lambda q \\ \sqrt{\lambda acq^{-1}}, -\sqrt{\lambda acq^{-1}}, \lambda c, a\lambda, \lambda, acq; q \end{matrix}\right] $$

$$ = \frac{(\lambda ac, \lambda; q)_\infty}{(\lambda c, \lambda a; q)_\infty}\left\{\frac{1}{1-}\frac{\lambda(1-ac)(1-a)(1-c)/(1-\lambda)(1-ac)(1-acq)}{\left(\frac{1-\lambda/q}{1-\lambda}\right)+}\right. $$

$$ \frac{\frac{\lambda}{q}(1-q)(1-cq)(1-aq)/(1-\lambda)(1-acq)(1-acq^2)}{1-} $$

$$ \left.\frac{\lambda q(1-acq)(1-aq)(1-cq)/(1-\lambda q)(1-acq^2)(1-acq^3)}{\left(\frac{1-\lambda/q}{1-\lambda q}\right)+}\cdots\right]. \qquad \text{.... (17)} $$

For $\lambda = q$, (4.17) yields :

$$ \frac{(aq, cq; q)_\infty}{(q, acq; q)_\infty} = \frac{1}{1-}\frac{q(1-ac)(1-a)(1-c)/(1-q)(1-ac)(1-acq)}{0+} $$

$$ \frac{(1-q)(1-aq)(1-cq)/(1-q)(1-acq)(1-acq^2)}{1-} $$

$$ \frac{q^2(1-acq)(1-aq)(1-cq)/(1-q^2)(1-acq^2)(1-acq^3)}{0+}\ \cdots . \qquad \text{... (18)} $$

Replacing a and c by q^a and q^c respectively, we get :

$$ \frac{(1-q^a)(1-q^c)}{(1-q)(1-q^{a+})}B_q(a, c) $$

$$ = 1 - \frac{q(1-q^a)(1-q^c)/(1-q^{(1+a+c)})(1-q)}{0+} $$

$$ \frac{(1-q^{a+1})(1-q^{c+1})/(1-q^{1+a+c})(1-q^{2+a+c})}{1-} $$

$$ \frac{q^2(1-q^{1+a+c})(1-q^{1+a})(1-q^{1+c})/(1-q^2)(1-q^{2+a+c})(1-q^{3+a+c})}{0+}\cdots \qquad \text{.... (19)} $$

As $q \to 1$, (4.19) gives :

$$ \left(\frac{ac}{a+c}\right)B(a, c) = \frac{\Gamma[a+1]\,\Gamma[c+1]}{\Gamma[1+a+c]} $$

$$= 1 - \frac{ac/(1+a+c)}{0+} \frac{(a+1)(c+1)/(1+a+c)(2+a+c)}{1-}$$

$$\frac{(1+a+c)(1+a)(1+c)/2(2+a+c)(3+a+c)}{0+} \dots . \qquad \dots (20)$$

Taking $a = x$, $c = 1 - x$ in (4.20) we get :

$$\frac{x(1-x)\,\pi}{\sin \pi x} = 1 - \frac{x(1-x)/2}{0+} \frac{(x+1)(2-x)/6}{1-} \frac{(1+x)(2-x)/12}{0+}$$

$$= 1 - \frac{x(1-x)}{0+} \frac{(x+1)(2-x)}{3-} \frac{(1+x)(2-x)}{0+} \frac{(2+x)(3-x)}{5-} \dots . \qquad \dots (21)$$

Taking $x = 1/2$ in (4.21) we get :

$$\frac{\pi}{4} = 1 - \frac{1/4}{0+} \frac{9/4}{3-} \frac{9/4}{0+} \frac{25/4}{5-} \dots$$

$$= 1 - \frac{1}{0+} \frac{9}{3-} \frac{9}{0+} \frac{25}{5-} \frac{25}{0+} \frac{49}{7-} \frac{49}{0+} \dots . \qquad .. (22)$$

If we take $\lambda \to q/ac$ in (4.16) we get :

$$1 + \sum_{n=1}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \frac{(q^2/ac)^n}{(q/ac;q)_n} \frac{(a, c, ac, 1/ac; q)_n}{(q/a, q/c, acq; q)_n} (1+q^n) \right\}^k$$

$$= \left\{ \frac{(q, q/ac; q)_\infty}{(q/a, q/c; q)_\infty} \right\}^k \left\{ \frac{1}{1-} \frac{\frac{q}{ac}(1-ac)(1-a)(1-c)/(1-q/ac)(1-ac)(1-acq)}{\left(\frac{1-1/ac}{1-q/ac}\right)+} \right.$$

$$\left. \frac{\frac{1}{ac}(1-q)(1-cq)(1-aq)/(1-q/ac)(1-acq)(1-acq^2)}{1-} \dots \right\}^k . \qquad \dots (23)$$

Taking $a, c \to \infty$ in (4.23), we obtain :

$$1 + \sum_{n=1}^{\infty} \left\{ (1+q^n)\, q^{n(3n-1)/2} \right\}^k = \left\{ (q;q)_\infty \right\}^k . \qquad \dots (24)$$

For $k = 1$, (4.24) yields Euler's identity :

$$\sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty . \qquad \dots (25)$$

For $k = 2$, (4.23) yields :

$$1 + \sum_{n=1}^{\infty} (1 + q^n)^2 q^{n(3n-1)} = \left\{ (q; q)_\infty \right\}^2. \qquad \text{... (26)}$$

By applying Jacobi's triple product identity, (4.26) can be written as:

$$(q^6; q^6)_\infty \left\{ (-q^3; q^6)_\infty^2 + (-q^2 - q^4 l\, q^6)_\infty \right\} = 1 + (q; q)_\infty^2. \qquad \text{... (27)}$$

A number of other interesting results can also be scored.

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