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# ON THE PRINCIPLE OF SUCCESSIVE APPROXIMATIONS

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In this paper we give some generalizations of Picard's principle of successive approximations for noncontinuous functions. The iterative sequence $(f^n x)_{n \in N}$ is here replaced by any sequence $(x_n)_{n \in N}$ with the property that both $x_n$ and $fx_n$ converge, as $n$ goes to infinity, to the same point of the underlying space. The first result deals with functions defined in metric spaces, then we generalize the theorem in the setting of Menger spaces and separated uniform spaces.

**Key Words : Successive Approximations; Noncontinuous Functions; Metric Spaces; Menger Spaces; Hausdorff Topological Spaces**

## 1. MAIN RESULTS

One of the most useful iterative techniques is the principle of successive approximations of Picard[2], which in the general context of Hausdorff topological spaces reads as follows (see Smart[3] page 1, Theorem 1.1.2): if $f$ is a continuous function from a Hausdorff topological space $X$ into itself and if $f^n x \to a \in X$ for $n \to +\infty$, then $fa = a$.

The aim of this paper is to generalize Picard's result (in particular topological spaces) for functions which may not be continuous.

*Definition* — Let $(X, \tau)$ be a topological space, $a \in X$ a cluster point of $X$ and $\varphi : X \to R^+$ a mapping; then $\varphi$ is said to be lower (upper) semicontinuous at $a$ if it satisfies

$$\varphi(a) \leq \liminf_{\xi \to a} \varphi(\xi) \left( \varphi(a) \geq \limsup_{\xi \to a} \varphi(\xi) \right).$$

The following result is essentially contained in Hicks and Rhoades[1].

***Theorem*** 1 — *Let* $(X, d)$ *be a metric space and* $f : X \to X$ *a function such that there exist two points* $a, x \in X$ *such that* $f^n x \to a$ *for* $n \to +\infty$; *then* $fa = a$ *if and only if the mapping* $\xi \mapsto d(\xi, f\xi)$ *is lower semicontinuous at* $a$.

Our first result is a simple generalization of the previous theorem in which the sequence $(f^n x)_{n \in N}$ is replaced by any sequence $(x_n)_{n \in N}$ with the property that both $x_n$ and $fx_n$ converge to $a$ as $n$ goes to infinity.

***Theorem*** 2 — *Let* $(X, d)$ *be a metric space and* $f : X \to X$ *a function such that there exist a point* $a \in X$ *and a sequence* $(x_n)_{n \in N} \subseteq X$ *such that* $x_n \to a$ *and* $fx_n \to a$ *for* $n \to +\infty$; *then* $fa = a$ *if and only if the mapping* $\xi \mapsto d(\xi, f\xi)$ *is lower semicontinuous at the limit point* $a$.

PROOF : Let us suppose that $fa = a$, then we have

$$d(a, fa) = 0 \leq \liminf_{\xi \to a} d(\xi, f\xi)$$

which gives the lower semicontinuity of $\xi \mapsto d(\xi, f\xi)$ at $a$.

Conversly, for the lower semicontinuity of $\xi \mapsto d(\xi, f\xi)$ at $a$, from the fact that $x_n, fx_n \to a$ for $n \to +\infty$ and from the triangular property for metric spaces, we have

$$d(a, fa) \leq \liminf_{\xi \to a} d(\xi, f\xi) \leq \lim_{n \to +\infty} d(x_n, fx_n) \leq \lim_{n \to +\infty} [d(x_n, a) + d(a, fx_n)] = 0$$

thus $fa = a$. □

*Example* — Let $X := \{2^k \mid k \in Z\} \cup \{0, +\infty\}$ with the metric induced from $\bar{R}$ and $f : X \to X$ the function defined by

$$fx \stackrel{def}{=} \begin{cases} x/2 \text{ if } x = 2^{2k} & k \in Z \\ 1/x \text{ if } x = 2^{2k+1} & k \in Z \\ 0 \quad \text{if } x = 0 \\ 1 \quad \text{if } x = +\infty \end{cases}$$

then, in this setting, all the hypotheses of Theorem 2 are satisfied, in fact taking $x_n = 2^{-2n}$ $n \in N$ one has $x_n \to 0$ and $fx_n = 2^{-2n-1} \to 0$ for $n \to +\infty, f0 = 0$ and obviously $\xi \mapsto d(\xi, f\xi)$ is lower semicontinuous at 0, but $f^n x$ does not converge for every $x \in X - \{0\}$, so that Theorem 1 cannot be applied.

We give now an analogue of Theorem 2 in the context of Menger and probabilistic metric spaces (*PM*-spaces). First of all we summarize the essential background material; for more details the reader is reminded to the monograph of Schweizer and Skalar[4].

A mapping $F : R \to R^+$ is called a distribution function if it is nondecreasing, left-continuous and such that $\inf F = 0$ and $\sup F = 1$. The set of all distribution functions will be denoted by $\Delta$.

A mapping $\tau : [0, 1]^2 \to [0, 1]$ is called a triangular norm if and only if it is symmetric, associative, nondecreasing in both places and such that for each $s \in [0, 1]$ one has $\tau(s, 1) = s$.

A *PM*-spaces is an ordered pair $(X, \Phi)$ where $X$ is an abstract set and $\Phi : X^2 \to \Delta$ is such that for each $(p, q) \in X^2$ $F_{p,q} := F(p, q)$ satisfies the following conditions:

(a) $F_{p,q}(t) = 1 \;\; \forall\, t > 0 \Leftrightarrow p = q$

(b) $F_{p,q}(0) = 0$

(c) $F_{p,q} = F_{q,p}$

(d) $F_{p,q}(t) = 1 \wedge F_{q,r}(s) = 1 \Rightarrow F_{p,r}(t+s) = 1.$

The triple $(X, \Phi, \tau)$, where $\tau$ is a triangular norm, will be called a Menger space if we also have

(e) $F_{p,r}(t+s) \geq \tau(F_{p,q}(t), F_{q,r}(s)).$

Given a point $p \in X$ and two positive real numbers $\varepsilon$ and $\lambda$, we can define the $(\varepsilon, \lambda)$-neighbourhood of $p$ $B_p(\varepsilon, \lambda)$ by the following

$$B_p(\varepsilon, \lambda) \stackrel{def}{=} \{q \in X \mid F_{p,q}(\varepsilon) > 1 - \lambda\}.$$

Thus we say that $x_n \to p$ for $n \to +\infty$ in the Menger space $(X, \Phi, \tau)$ if and only if for each $\varepsilon, \lambda > 0$ there exists a $\nu \in N$ such that for each $n > \nu$ one has $x_n \in B_p(\varepsilon, \lambda)$, that is $F_{p, x_n}(\varepsilon) > 1 - \lambda$.

***Theorem* 3** — *Let $(X, \Phi, \tau)$ be a Menger space, with the triangular norm $\tau$ such that $\sup_{s<1} \tau(s, s) = 1$, and $f : X \to X$ a function such that there exist a point $a \in X$ and a sequence $(x_n)_{n \in N} \subseteq X$ such that $x_n \to a$ and $fx_n \to a$ for $n \to +\infty$; then $fa = a$ if and only if for each $t > 0$ the mapping $\varphi_t : X \to R^+$ defined by $\varphi_t(x) := F_{x, fx}(t)$ is upper semicontinuous at the point $a$.*

PROOF : Let us suppose that $fa = a$, thus from property (a) for each $t > 0$ we have

$$F_{a, fa}(t) = 1 \geq \limsup_{\xi \to a} F_{\xi, f\xi}(t)$$

which gives the upper semicontinuity of $\varphi_t$ for each $t > 0$ at $a$.

Conversely if this happens, from the fact that $x_n \to a$ for $n \to +\infty$ one has

$$F_{a, fa}(t) \geq \limsup_{\xi \to a} F_{\xi, f\xi}(t) \geq \lim_{n \to +\infty} F_{x_n, fx_n}(t) \quad \forall\, t > 0$$

but now $x_n, fx_n \to a$ for $n \to +\infty$ means that for each $\varepsilon, \lambda > 0$ there exists a $\nu = \nu(\varepsilon, \lambda) \in N$ such that for each $n > \nu$ one has $F_{x_n, a}(\varepsilon), F_{a, fx_n}(\varepsilon) > 1 - \lambda$, that is

$$\lim_{n \to +\infty} F_{x_n, a}(\varepsilon) = \lim_{n \to +\infty} F_{a, fx_n}(\varepsilon) = 1$$

thus, by property $(e)$ with $\varepsilon = t/2$, by the properties of a triangular norm and from the fact that $\sup_{s<1} \tau(s, s) = 1$ one has

$$1 \geq F_{x_n, fx_n}(t) \geq \tau(F_{x_n, a}(t/2), F_{a, fx_n}(t/2)) \stackrel{n \to +\infty}{\longrightarrow} 1$$

that is for each $t > 0$ $\lim_{n \to +\infty} F_{x_n, fx_n}(t) = 1$ and thus for each $t > 0$ $F_{a, fa}(t) = 1$ i.e. $fa = a$. □

*Remark* : From the proof of Theorem 3 it is evident that one can replace the triangular norm with a genetic function $\tau : [0, 1]^2 \to [0, 1]$ continuous in (1.1) and such that $\tau(1, 1) = 1$, without changing the conclusion of the theorem.

Our last theorem deals with successive approximations living in separated uniform spaces; as for the previous result we give here the essential definitions and notations.

We call $(X, \mathcal{U})$ a separated uniform space if $X$ is an abstract set and $\mathcal{U} \subseteq \mathcal{P}(X^2)$ is such that

$(i)$ $U \in \mathcal{U} \Rightarrow \Delta \subseteq U$ $(\Delta := \{(x, x) \mid x \in X\})$

$(ii)$ $U_1, U_2 \in \mathcal{U} \Rightarrow \exists\, V \in \mathcal{U};\ V \subseteq U_1 \cap U_2$

$(iii)$ $U \in \mathcal{U} \Rightarrow \exists\, V \in \mathcal{U};\ V \subseteq U^{-1}$

$(U^{-1} := \{(x, y) \in X \times X \mid (y, x) \in U\})$

(*iv*) $U \in \mathcal{U} \Rightarrow \exists V \in \mathcal{U}; V \circ V \subseteq U$

$(U \circ V := \{(x, z) \in X \times X \mid \exists y \in X; (x, y) \in U \wedge (y, z) \in V\})$

(*v*) $\bigcap_{U \in \mathcal{U}} U = \Delta$.

We say that a function $f$, defined from a separated uniform space $(X, \mathcal{U})$ into itself, satisfies the (ul)-condition at a point $a \in X$ if and only if one has: if there is a $V \in \mathcal{U}$ such that for all $U \in \mathcal{U}$ for which there exists a $\xi_U \in X$ with $(a, \xi_U) \in U$ and $(\xi_U, f\xi_U) \in V$, then $(a, fa) \in U \circ V$.

***Theorem*** 4 — *Let $(X, \mathcal{U})$ be a separated uniform space, and $f: X \to X$ a function such that there exist a point $a \in X$ and a sequence $(x_n)_{n \in N} \subseteq X$ such that $x_n \to a$ and $fx_n \to a$ for $n \to +\infty$; then $fa = a$ if and only if $f$ satisfies the (ul)-condition at the limit point $a$.*

PROOF : Let us suppose that $fa = a$ so that $(a, fa) \in \Delta \subseteq U \circ V$ for each $U, V \in \mathcal{U}$; thus f trivially satisfies the (ul)-condition at $a$.

Conversly from the fact that $\lim_{n \to \infty} x_n = a$, given an arbitrary $W \in \mathcal{U}$, one has that for each $V \in \mathcal{U}$ such that $V \circ V \subseteq W$ there exists a $n_V \in N$ such that for each $n > n_V$ one has $(a, x_n) \in V$; further from the fact that $\lim_{n \to \infty} fx_n = a$, given an arbitrary $W \in \mathcal{U}$ and an arbitrary $V \in \mathcal{U}$ such that $V \circ V \subseteq W$, for each $U \in \mathcal{U}$ such that $U \circ U \subseteq V$ there exists a $n_U \in N$ such that for each $n > n_U$ one has $(x_n, a) \in U$ and $(a, fx_n) \in U$, so that $(x_n, fx_n) \in U \circ U \subseteq V$; thus $(a, x_n), (x_n, fx_n) \in V$ and for the (ul)-condition at $a$

$$(a, fa) \in V \circ V \subseteq W \text{ for each } W \in \mathcal{U}$$

that is
$$(a, fa) \in \bigcap_{W \in \mathcal{U}} W = \Delta \text{, i.e. } fa = a.$$
□

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