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KELVIN-HELMHOLTZ INSTABILITY IN THE NEIGHBOURHOOD OF THE MINIMUM OF THE NEUTRAL CURVE IN ELECTROHYDRODYNAMICS

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The nonlinear instability for two streaming superposed fluids in the presence of a normal electric field is investigated taking into account the spatial as well as temporal effects by using the method of multiple scales. A generalised formulation of the evolution equation governing the amplitude is developed when the difference between the velocities of the two fluids is less than or equal to the critical velocity. It is shown that for the supercritical values of the velocity difference, the instability is governed by a nonlinear Klein-Gordon equation. Stability criteria are derived from this equation. It is demonstrated that the electric field decreases the critical velocity at which the instability sets in.

Key Words : Kelvin-Helmholtz Instability; Neutral Curve; Electrohydrodynamics; Nonlinear Klein-Gordon Equation

1. INTRODUCTION

The study of nonlinear dispersive waves, such as the works of Lighthill[11], Karpman[9], Whitham[8], and others, has been a subject of major interest in recent years. It was reported experimentally by Gross and Porter[4] that instability occurs when a thermally and gravitationally stable stratification of a fluid is subject to an electric field. This instability was explained on the hypothesis that when a non-uniform field is applied to an inhomogeneous fluid having variable dielectric constant, regions of lower field strength will experience a force directed towards regions of higher field strength. Thus instablity may arise due to variation of the dielectric constant because of heating or any other source of inhomogeneity. Later, it was found that the temperature dependence of the dielectric constant has no significant effect on the fluid layer, and stabilizing or destabilizing effects of the electric field depend on whether the conductivity is a linear function of temperature or a quadratic function.

The character of equilibrium of superposed fluids in relative motion has been the subject matter of great interest ever since the pioneering work by Helmholtz and Kelvin (see : Chandrasekhar[1]). The nonlinear : Kelvin-Helmholtz instability of ideal fluids has been investigated by Drazin[6], Nayfeh and Saric[7] and Weissman[12] taking into account surface tension and a gravitational body force. Nayfeh and Saric found that the effect of the nonlinearity on the stability of the flow

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depends on density ratio and the surface tension. Weissman examined the classical model of Lord Kelvin with allowance for small, but finite, disturbances and for spatial as well as temporal development and obtained corrections to the results of Nayfeh and Saric.

The main difference between the Kelvin-Helmholtz instability and the Rayleigh-Taylor instability is the inclusion of the term $V \cdot \nabla V$ in the perturbation equation, which is a nonlinear term. This led Thorpe[19] to find in his experiments' results not predicted by the linear theory; he observed that streaming plays a dual role in the stability criterion. In the meantime, if a normal electric field is applied to a Rayleigh-Taylor surface such that there are no surface charges on the interface, one finds that nonlinear results vary considerably from those of the linear theory and that the electric field plays a dual role in the stability criterion. Moreover, a role is played by the dielectric constants. It is important to know which of the fluids has the larger dielectric constant. We aim here to couple the two cases, namely the *K-H* flow with a normal field, to examine how these nonlinearities interact. Of principal concern is the neighbourhood of the critical point for instability, where weakly nonlinear solutions can be found for arbitrary initial conditions.

In this presentation, we examime how a continuous band width of the modes affects the description of the post critical normal field instability on charge-free surface separating two semi-infinite dielectric fluids. A bifurcation analysis is presented using the method of multiple scales. We are primarily interested in the weakly nonlinear solutions in the neighbourhood of the critical point for instability i.e., when the velocity difference V is slightly greater than the critical velocity V_c. In Section 2, we formulate the problem, give the basic equations, review the linear theory governing the wave motion on the surface of two superposed electric fluids and obtain the critical velocity V_c and the critical wave number k_c. In Section 3, we show that the evolution of the small but finite amplitude wave-packet is governed by the nonlinear Klein-Gordon equation. Depending upon the value of electric field strength E, the dielectric constants and the density ratio of the media involved, this equation gives exponential decay, subcritical and explosive instabilities if only temporal variations are taken into consideration. The nonlinear Klein-Gordon equation has a wave train solution in terms of the Jacobi elliptic functions from which we have obtained the solitary wave, kink, and antikink solutions.

2. FORMULATION OF THE PROBLEM

We consider two semi-infinite dielectric fluids separated by the plane $y = 0$. The fluids are assumed to be incompressible, inviscid with densities $\rho^{(1)}, \rho^{(2)}$ and streaming with velocities $U^{(1)}$ and $U^{(2)}$ along the positive x-direction. The superscripts 1 and 2 refer to the region $y < 0$ and $y > 0$ respectively; $e^{(1)}$ and $e^{(2)}$ are the dielectric constants of the fluids. The gravitational force per unit mass is $g(0, -1, 0)$ and both the fluids are subject to a constant electric field in the y-direction $E^{(1)}$ and $E^{(2)}$. The interface is taken to be $S(x, t) = y - \eta(x, t) = 0$, where $\eta(x, t)$ is the surface elevation of the free surface from the unperturbed level. The motion is assumed to be irrotational having velocity potential ϕ and the lighter fluid is overlying the heavier fluid. We shall assume that there are no surface charges at the surface of separation in the equilibrium state, and therefore the electric displacement is continuous at the interface, i.e.,

$$e^{(1)} E^{(1)} = e^{(2)} E^{(2)}. \qquad \ldots (1)$$

In our analysis the various quantities are nondimensionalized using the characteristic length $[\sigma/\rho^{(1)} g]^{1/2}$ and the characteristic time $[L/g]^{1/2}$, where σ is the surface tension and g is the

acceleration due to gravity acting in negative y-direction. The velocity potential $\phi'^{(j)}$ describing the motion can be expressed in a nondimensional form as

$$\phi'^{(j)} = (gL^3)^{-1/2} [V^{(j)} x + \phi^{(j)} (x, y, t)], \quad j = 1, 2, \qquad \ldots (2)$$

where $\phi^{(j)} (x, y, t)$ are the perturbation potential and

$$V^{(j)} = U^{(j)} (Lg)^{-1/2}. \qquad \ldots (3)$$

The velocity potential ϕ satisfies the Laplace's equation

$$\nabla^2 \phi^{(2)} = 0, \ \eta(x, t) < y < \infty \qquad \ldots (4)$$

and

$$\nabla^2 \phi^{(1)} = 0, \ -\infty < y < \eta(x, t). \qquad \ldots (5)$$

Since the motion must vanish away from the interface, the solutions of ϕ have to satisfy, at large distance, the conditions:

$$| \nabla \phi^{(2)} | \to 0 \quad \text{as} \quad y \to \infty; \qquad \ldots (6)$$

and

$$| \nabla \phi^{(1)} | \to 0 \quad \text{as} \quad y \to -\infty. \qquad \ldots (7)$$

The displacement of the interface at time t is assumed to be of the form

$$y = \eta(x, t). \qquad \ldots (8)$$

The unit normal $\tilde{N}$ to the surface is then given by

$$\tilde{N} = \frac{\nabla F}{| \nabla F |} = -\eta_x \left(\eta_x^2 + 1 \right)^{-1/2} \tilde{e}_x + \left(\eta_x^2 + 1 \right)^{-1/2} \tilde{e}_y, \qquad \ldots (9)$$

where $F = 0$ is the equation of the surface of separation. The condition that the interface is moving with the fluid leads to

$$\phi_y^{(j)} = \eta_t + V^{(j)} \eta_x + \eta_x \phi_x^{(j)} \quad \text{at } y = \eta(x, t), \ j = 1, 2. \qquad \ldots (10)$$

In electrohydrodynamics, it is generally assumed that the quasi-static approximation is valid and the electric field is irrotational having electrostatic potential ψ and thus we introduce the electrostatic potentials $\psi^{(j)} (x, y, t)$ ($j = 1, 2$ in the appropriate region) such that

$$\tilde{E}^{(j)} = E^{(j)} \tilde{e}_y - \nabla \psi^{(j)}. \qquad \ldots (11)$$

The differential equation satisfied by $\psi^{(j)}(x, y, t)$ is Laplace's equation

$$\nabla^2 \psi^{(j)} = 0, \quad j = 1, 2, \qquad \ldots (12)$$

along with the boundary condition that the tangential component of the electric field is continuous at the interface, so that

$$\eta_x \left[\!\left[\psi_y \right]\!\right] + \left[\!\left[\psi_x \right]\!\right] = \eta_x \left[\!\left[E \right]\!\right] \text{ at } y = \eta(x, t), \qquad \ldots (13)$$

where $[\, \nu \,] = \nu^{(1)} - \nu^{(2)}$, represents the jump in ν across the interface $y = 0$.

Since there are no surface charges at the interface $y = \eta$, the normal electric displacement is continuous at the interface. Therefore

$$\eta_x \left[\!\left[e\psi_x \right]\!\right] - \left[\!\left[e\psi_y \right]\!\right] = 0 \text{ at } y = \eta(x, t). \qquad \ldots (14)$$

The electric stress tensor is given by

$$\Pi_{ij} = -\Pi\delta_{ij} + eE_iE_j,$$

where $\Pi = p - \frac{1}{2} eE^2$, p is pressure, e is permitivity, and E is the electric field. The normal hydrodynamical stress is balanced by the normal electric stress. The balance condition is then

$$\begin{aligned}
&\phi_t^{(1)} + V^{(1)} \phi_x^{(1)} + \frac{1}{2}\left(\phi_x^{(1)^2} + \phi_y^{(1)^2} \right) + (1 - \rho)\eta \\
&= \rho\left[\phi_t^{(2)} + V^{(2)} \phi_x^{(2)} + \frac{1}{2}\left(\phi_x^{(2)^2} + \phi_y^{(2)^2} \right) \right] + \eta_{xx}\left(1 + \eta_x^2 \right)^{-3/2} - \frac{1}{2}\left[\!\left[e\psi_x^2 \right]\!\right] \\
&\quad - \left[\!\left[eE\,\psi_y \right]\!\right] + \frac{1}{2}\left[\!\left[e\psi_y^2 \right]\!\right] + 2\eta_x \left[\!\left[eE\,\psi_x \right]\!\right] - 2\eta_x \left[\!\left[e\psi_x\,\psi_y \right]\!\right] + \eta_x^2 \left[\!\left[e\psi_x^2 \right]\!\right] \\
&\quad - \eta_x^2 \left[\!\left[eE^2 \right]\!\right] + 2\eta_x^2 \left[\!\left[eE\,\psi_y \right]\!\right] - \eta_x^2 \left[\!\left[e\psi_y^2 \right]\!\right] - 2\eta_x^3 \left[\!\left[eE\,\psi_x \right]\!\right] \\
&\quad + 2\eta_x^3 \left[\!\left[e\psi_x\,\psi_y \right]\!\right] + O(\eta_x^4) \text{ at } y = \eta(x, t), \qquad \ldots (16)
\end{aligned}$$

where $\rho = \rho^{(2)}/\rho^{(1)}$.

To describe the nonlinear interactions of small but finite amplitude wave, we use the method of multiple scales as formulated by Weissman[12] and Murakami[16]. We wish to find solutions in the neighbourhood of the minimum of the neutral curve where the linear growth rate of instability is small. We wish to describe the mode which is being modulated slowly in time and space. Thus, to investigate the effects of spatial as well as temporal modulation in the neighbourhood of (k_c, V_c), we employ the method of multiple scales, and introduce spatial and temporal scales :

$$x_n = \varepsilon^n x, \; t_n = \varepsilon^n t, \; n = 0, 1, 2, \ldots, \qquad \ldots (17)$$

where the velocity difference parameter $\varepsilon = \left(\frac{V}{V_c} - 1 \right)^{1/2}$, assumed small, defines the departure of the system from the bifurcation value V_c (See : Eckhaus[3]. We write for any variable

$$f(x, y, t) = \sum_{n=1}^{3} \varepsilon^n f_n + O(\varepsilon^4), \qquad \ldots (18)$$

where f is any of the physical variables and is a function of slow scale variables in space and time. We allow various perturbation quantities η_n, ϕ_n, ψ_n to depend in general on all these variables. While writing the expansion for η, it will be noted that η depends only on x and t and not on y. The differential operators in various equations are expanded as follows :

$$\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \alpha_0} + \varepsilon \frac{\partial}{\partial \alpha_1} + \varepsilon^2 \frac{\partial}{\partial \alpha_2} + \ldots, \qquad \ldots (19)$$

where α is any of the variables x or t.

To evaluate the boundary conditions (10), (13), (14), and (16), we use Maclaurin's series at $y = 0$ for the quantities involved. The approximate system of equations can be obtained by substituting the expansions as defined into the set of eqs. (4), (5), (10), (12)-(14), and (16). This leads to successive higher order perturbation equations (see *Appendix*). It is sufficient to consider only the solution up to $O(\varepsilon^3)$ as far as the lowest significant approximation is concerned.

The progressive wave solution of the first-order system governed by equations (A.2)-(A.6) is obtained as

$$\eta_1 = A \exp [i\theta_0] + \text{c.c.}, \qquad \ldots (20)$$

$$\phi_1^{(1)} = i \left(V^{(1)} - \frac{\omega}{k} \right) A \exp [i\theta_0 + ky] + \text{c.c.}, \qquad \ldots (21)$$

$$\phi_1^{(2)} = -i \left(V^{(2)} - \frac{\omega}{k} \right) A \exp [i\theta_0 - ky] + \text{c.c.}, \qquad \ldots (22)$$

$$\psi_1^{(1)} = WE^{(1)} A \exp [i\theta_0 + ky] + \text{c.c.} \qquad \ldots (23)$$

and

$$\psi_1^{(2)} = -WE^{(2)} A \exp [i\theta_0 - ky] + \text{c.c.}, \qquad \ldots (24)$$

where c.c. indicates the complex conjugate of all preceeding terms, $W = \frac{(e^{(2)} - e^{(1)})}{(e^{(2)} + e^{(1)})}$, $A(x_1, x_2, t_1, t_2)$ is the complex amplitude of the plane wave, and $\theta_0 = kx_0 - \omega t_0$ is the phase of the travelling wave with k and ω being the wavenumber and frequency of the center of the wave packet, respectively.

The solutions of the first order problem are valid provided ω satisfies the dispersion relation

$$D(\omega, k) = k \left(V^{(1)} - \frac{\omega}{k} \right)^2 + k\rho \left(V^{(2)} - \frac{\omega}{k} \right)^2 - [(1 - \rho) + \delta_0 k + k^2] = 0, \qquad \ldots (25)$$

where

$$\delta_0 = -W(e^{(2)} - e^{(1)})\, E^{(1)}\, E^{(2)}. \qquad \ldots (26)$$

or

$$\omega = k\frac{V^{(1)} + \rho V^{(2)}}{1+\rho} \pm \frac{1}{(1+p)^{1/2}}\left[k^3 + k^2(\delta_0 + \delta_1) + k(1-p)\right]^{1/2}, \qquad \ldots (27)$$

where

$$\delta_1 = -\rho\frac{(V^{(2)} - V^{(1)})^2}{(1+\rho)}. \qquad \ldots (28)$$

The interface of the two fluids is stable when ω is real and is unstable when ω is complex. Thus, when

$$k^2 + k[\delta_0 + \delta_1] + (1-\rho)$$

> 0, the flow is linearly stable,

< 0, the flow is linearly unstable,

$= 0$, the flow is neutrally stable. $\qquad \ldots (29)$

Thus the interface is linearly stable or unstable depending on whether k is larger or smaller than

$$k_{CE} = (\rho - 1)^{1/2}[\sinh(\theta_E) + \cosh(\theta_E)], \qquad \ldots (30)$$

where

$$\sinh(\theta_E) = \frac{\alpha_E}{2}(\rho - 1)^{1/2}, \qquad \ldots (31)$$

and

$$\alpha_E = -(\delta_0 + \delta_1). \qquad \ldots (32)$$

Thus, the critical wave number k_{CE} is the linear electrohydrodynamic cutoff wave number separating stable from unstable regions.

For simplicity, we assume that the lower fluid is at rest ($V^{(1)} = 0$) and the upper fluid is moving with a uniform velocity $V(V^{(2)} = V)$. The dispersion relation (25), leads to the following relations :

$$D_\omega = \frac{\partial D}{\partial \omega} = 2\frac{\omega}{k}(1+\rho) - 2(V^{(1)} + \rho V^{(2)}), \qquad \ldots (33)$$

$$D_{\omega\omega} = \frac{\partial^2 D}{\partial \omega^2} = \frac{2}{k}(1+\rho), \qquad \ldots (34)$$

$$D_k = \frac{\partial D}{\partial k}(V^{(1)^2} + \rho V^{(2)^2}) - \frac{\omega^2}{k^2}(1+\rho) - \delta_0 - 2k, \quad \ldots (35)$$

$$D_{kk} = \frac{\partial^2 D}{\partial k^2} = 2\frac{\omega^2}{k^3}(1+\rho) - 2, \quad \ldots (36)$$

$$D_{\omega k} = \frac{\partial^2 D}{\partial \omega} = -2\frac{\omega}{k^2}(1+\rho) \quad \ldots (37)$$

and

$$D_V = \frac{\partial D}{\partial V} = 2k\rho\,(V - \omega/k). \quad \ldots (38)$$

To obtain the Neutral Stability Curve, we note that without dissipation, there must be both growing and decaying modes in the unstable region i.e., modes with complex conjugate values for ω (assuming k real). On the neutral surface, the complex conjugate pair coalesces (i.e, they grow and decay together and unite into one) and hence the dispersion relation $D(\omega, k) = 0$ from (25) must yield a double root for ω (real) and obviously this can only happen if the first derivative of D with respect to ω vanishes, i.e. if $D_\omega = \frac{\partial D}{\partial \omega} = 0$. Thus, on the marginally neutral curve, $D_\omega = 0$ implies that

$$\omega = k\frac{\rho V}{1+\rho}, \quad \ldots (39)$$

and hence from the dispersion relation (27), the points of marginal stability lie on the 'neutral curve',

$$V^2 = V_m^2(k) = \left[\frac{1+\rho}{\rho}\left\{k + \frac{(1-\rho)}{k} + \delta_0\right\}\right], \quad \ldots (40)$$

and the neutral stability curve provides the critical value of velocity for the onset of instability. Thus, the critical velocity V_c occurs at $k = k_c = [1-\rho]^{1/2}$ and is given by

$$V_c = \left[\frac{1+\rho}{\rho}\left\{2[1-\rho]^{1/2} + \delta_0\right\}\right]^{1/2}. \quad \ldots (41)$$

Fig. 1 shows the neutral curves for V^2 against the wave number k [from (40)] for different values of $\rho^{(1)}$, $\rho^{(2)}$, $e^{(1)}$, $e^{(2)}$, $E^{(1)}$ and $E^{(2)}$. It is clear from the figure that the electric field decreases the critical velocity at which the instability sets in, while the critical wave number remains constant for a given density ratio.

It is obvious from (25) and (40) that the system is unstable for $V \geq V_c$. For values of $V < V_c$, all wave solutions are stable.

Let us now proceed to find solutions in the neighbourhood of the minimum of the neutral curve, the critical point (k_c, V_c) for the instability. We shall study the motion when

$$V = V_c(1 + \varepsilon^2\mu), \mu = \pm 1, \varepsilon << 1 \quad \ldots (42)$$

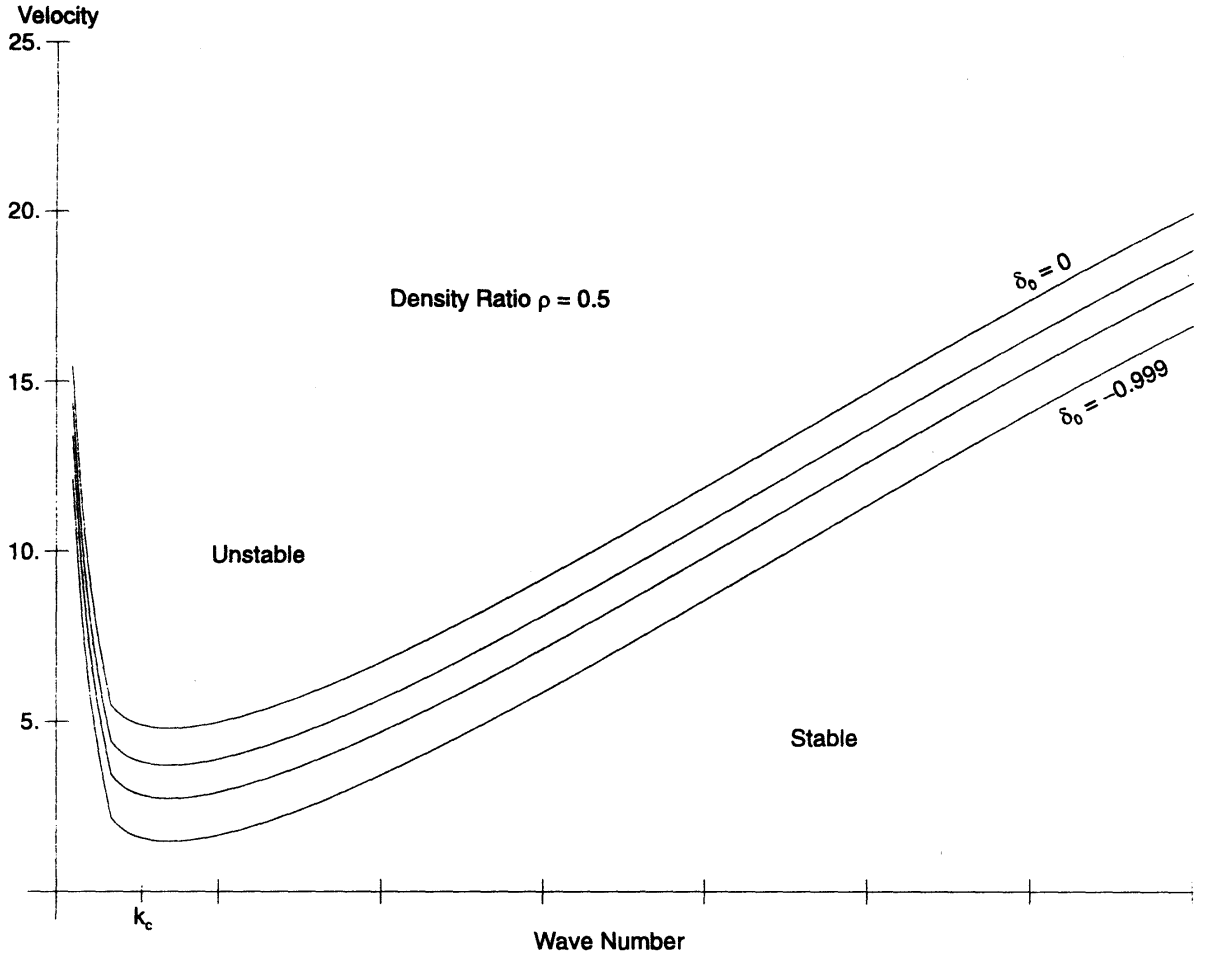

FIG. 1. Neutral Curve

The neutral curve for stability is parabolic in the close vicinity of (k_c, V_c). For any arbitrary initial data, the waves inside the unstable region with large growth rate will draw energy from the electric field system, thus giving rise to instability.

The perturbation expansion for surface elevation can be expressed as :

$$\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + \ldots . \qquad \ldots (43)$$

where

$$\eta_1 = A(x_1, x_2, t_1, t_2) \exp [i(k_c x_0 - \omega_c t_0)] + \text{ c.c.} \qquad \ldots (44)$$

Here, ω_c is the real part of the frequency corresponding to the critical point and is given by (39).

3. THE EVOLUTION EQUATIONS

If we carry out the problem to the second-order equations (A.7)-(A-11), one may substitute the solutions of the first-order problem (20)-(24) into the second-order one and solve the resulting

equations. The solutions of the second order problem are given by

$$\eta_2 = \lambda [A^2 \exp(2i\theta_0] + C \cdot C., \quad \ldots (45)$$

$$\phi_2^{(1)} = \frac{1}{k}\left[\frac{\partial A}{\partial t_1} + \frac{\partial A}{\partial x_1}\left\{V^{(1)} ky + \frac{\omega}{k}(1 - ky)\right\}\right] \exp[i\theta_0 + ky]$$

$$+ i\left(V^{(1)} - \frac{\omega}{k}\right)(\lambda - k) A^2 \exp[2i\theta_0 + 2ky] + \text{c.c.}, \quad \ldots (46)$$

$$\phi_2^{(2)} = -\frac{1}{k}\left[\frac{\partial A}{\partial t_1} + \frac{\partial A}{\partial x_1}\left\{-V^{(2)} ky + \frac{\omega}{k}(1 + ky)\right\}\right] \exp[i\theta_0 - ky]$$

$$- i\left(V^{(1)} - \frac{\omega}{k}\right)(\lambda - k) A^2 \exp[2i\theta_0 - 2ky] + \text{c.c.}, \quad \ldots (47)$$

$$\psi_2^{(1)} = -iWE^{(1)} \frac{\partial A}{\partial x_1} y \exp[i\theta_0 - ky] + WE^{(1)} (\lambda - k) A^2 \exp[2i\theta_0 - 2ky] + \text{c.c.} \quad \ldots (48)$$

and

$$\psi_2^{(2)} = -iWE^{(2)} \frac{\partial A}{\partial x_1} Y \exp[i\theta_0 + ky] + WE^{(2)} (\lambda + k) A^2 \exp[2i\theta_0 + 2ky] + \text{c.c.}, \quad \ldots (49)$$

where

$$\lambda = \left[k^2\left(V^{(1)} - \frac{\omega}{k}\right)^2 - \rho k^2\left(V^{(2)} - \frac{\omega}{k}\right)^2 + k^2 E^{(2)} E^{(1)} (e^{(2)} - e^{(1)}) W^2\right] / (1 - \rho - 2k^2). \quad \ldots (50)$$

The point $k = [(1 - \rho)/2]^{1/2}$ is due to the occurrence of second harmonic resonance and has been investigated by Singla *et al.*[20]. These solutions of the second order problem yield the following solvability condition for η_2 :

$$\left[-2(1 + \rho)\frac{\omega}{k} + 2(V^{(1)} + \rho V^{(2)})\right]\frac{\partial A}{\partial t_1}$$

$$+ \left[-2k - \frac{\omega^2}{k^2}(1 + \rho) - \delta_0 + \left(V^{(1)^2} + \rho V^{(2)^2}\right)\right]\frac{\partial A}{\partial x_1} = 0. \quad \ldots (51)$$

i.e.,

$$-\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial t_1} + \frac{\partial D}{\partial k}\frac{\partial A}{\partial x_1} = 0. \quad \ldots (52)$$

We now substitute the first order solutions (22)-(26) and second order solutions (45)-(49) into the third order problem governed by equations (A.12)-(A.16) and obtain the following equations:

$$\frac{\partial^2 \phi_3^{(1)}}{\partial x_0^2}+\frac{\partial^2 \phi_3^{(1)}}{\partial y^2}=\left[i\frac{\partial^2 A}{\partial x_1^2}\left\{ -\left(V^{(1)}+\frac{\omega}{k}\right)-2k\left(V^{(1)}-\frac{\omega}{k}\right)y\right\}-2i\frac{\partial^2 A}{\partial x_1\,\partial t_1}\right.$$

$$\left. +\ 2k\left(V^{(1)}-\frac{\omega}{k}\right)\frac{\partial A}{\partial x_2}\right]\exp\left(i\theta_0+ky\right)\ +\ \ldots, \qquad \ldots\ (53)$$

$$\frac{\partial^2 \phi_3^{(2)}}{\partial x_0^2}+\frac{\partial^2 \phi_3^{(2)}}{\partial y^2}\ =\left[i\frac{\partial^2 A}{\partial x_1^2}\left\{\left(V^{(2)}+\frac{\omega}{k}\right)-2k\left(V^{(2)}-\frac{\omega}{k}\right)y\right\}+2i\frac{\partial^2 A}{\partial x_1\,\partial t_1}\right.$$

$$\left. -\ 2k\left(V^{(1)}-\frac{\omega}{k}\right)\frac{\partial A}{\partial x_2}\right]\exp\left(i\theta_0-ky\right)\ +\ \ldots, \qquad \ldots\ (54)$$

$$\frac{\partial^2 \psi_3^{(1)}}{\partial x_0^2}+\frac{\partial^2 \psi_3^{(1)}}{\partial y^2}=-WE^{(1)}\left[(1+2ky)\frac{\partial^2 A}{\partial x_1^2}+2ik\frac{\partial A}{\partial x_2}\right]\exp\left(i\theta_0+py\right)\ +\ \ldots \qquad \ldots\ (55)$$

and

$$\frac{\partial^2 \psi_3^{(2)}}{\partial x_0^2}+\frac{\partial^2 \psi_3^{(2)}}{\partial y^2}=WE^{(2)}\left[(1-2ky)\frac{\partial^2 A}{\partial x_1^2}+2ik\frac{\partial A}{\partial x_2}\right]\exp\left(i\theta_0-ky\right)\ +\ \ldots\ ,\ldots\ (56)$$

with the boundary conditions at $y = 0$:

$$\frac{\partial \phi_3^{(1)}}{\partial y}-\frac{\partial \eta_3}{\partial t_0}-V^{(1)}\frac{\partial \eta_3}{\partial x_0}$$

$$=\left[\frac{\partial A}{\partial t_2}+V^{(1)}\frac{\partial A}{\partial x_2}+A^2\bar{A}\left\{\frac{1}{2}ik^2\left(V^{(1)}-\frac{\omega}{k}\right)(-6\lambda+k)\right\}\right]\exp\left(i\theta_0\right)+\ldots\ , \qquad \ldots\ (57)$$

$$\frac{\partial \phi_3^{(2)}}{\partial y}-\frac{\partial \eta_3}{\partial t_0}-V^{(2)}\frac{\partial \eta_3}{\partial x_0}$$

$$=\left[\frac{\partial A}{\partial t_2}+V^{(2)}\frac{\partial A}{\partial x_2}+A^2\bar{A}\left\{\frac{1}{2}ik^2\left(V^{(2)}-\frac{\omega}{k}\right)(6\lambda+k)\right\}\right]\exp\left(i\theta_0\right)+\ \ldots, \qquad \ldots\ (58)$$

$$\frac{\partial \psi_3^{(1)}}{\partial x_0}-E^{(1)}\frac{\partial \eta_3}{\partial x_0}-\frac{\partial \psi_3^{(2)}}{\partial x_0}+E^{(2)}\frac{\partial \eta_3}{\partial x_0}$$

$$= \left[\frac{1}{2} ik^2 W \left\{ E^{(2)} (6\lambda + k) - E^{(1)} (6\lambda - k) \right\} \right] A^2 \bar{A} \exp (i\theta_0) + \ldots\ldots\ldots , \quad \ldots (59)$$

$$e^{(1)} \frac{\partial \psi_3^{(1)}}{\partial y} - e^{(2)} \frac{\partial \psi_3^{(2)}}{\partial y} = [-2\lambda k^2 e^{(1)} E^{(1)} W] A^2 \bar{A} \exp (i\theta_0) + \ldots , \quad \ldots (60)$$

$$(1-\rho)\eta_3 + \frac{\partial \phi_3^{(1)}}{\partial t_0} - \rho \frac{\partial \phi_3^{(2)}}{\partial t_0} - e^{(1)} E^{(1)} \frac{\partial \psi_3^{(1)}}{\partial y}$$

$$+ e^{(2)} E^{(2)} \frac{\partial \psi_3^{(2)}}{\partial y} - \frac{\partial^2 \eta_3}{\partial x_0^2} + V^{(1)} \frac{\partial \phi_3^{(1)}}{\partial x_0} - \rho V^{(2)} \frac{\partial \phi_3^{(2)}}{\partial x_0} = \left[\frac{\partial^2 A}{\partial t_1^2} \left\{ -\frac{1}{k} (1+\rho) \right\} \right.$$

$$+ \frac{\partial^2 A}{\partial x_1^2} \left\{ 1 - \frac{\omega}{k^2} (V^{(1)} + \rho V^{(2)}) \right\} + \frac{\partial^2 A}{\partial x_1 \partial t_1} \left\{ -\frac{\omega}{k^2} (1+\rho) - \frac{1}{k} (V^{(1)} + \rho V^{(2)}) \right\}$$

$$+ \frac{\partial A}{\partial t_2} \left\{ i \frac{\omega}{k} (1+\rho) - i (V^{(1)} + \rho V^{(2)}) \right\} + i \cdot \frac{\partial A}{\partial x_2}$$

$$\left\{ 2k - (V^{(1)^2} + \rho V^{(2)^2}) + \frac{\omega}{k} (V^{(1)} + \rho V^{(2)}) \right\}$$

$$+ A^2 \bar{A} \left\{ \lambda k^2 \left(\left(V^{(1)} - \frac{\omega}{k} \right)^2 - \rho \left(V^{(2)} - \frac{\omega}{k} \right)^2 \right) \right.$$

$$- \frac{5}{2} k^3 \left(\left(V^{(1)} - \frac{\omega}{k} \right)^2 + \rho \left(V^{(2)} - \frac{\omega}{k} \right)^2 \right)$$

$$+ e^{(1)} E^{(1)^2} W \left(-\frac{5}{2} k^3 - 3\lambda k^2 + 4\lambda k^2 + 4\lambda k^2 W^{-1} \right)$$

$$\left. + e^{(2)} E^{(2)^2} W \left(\frac{5}{2} k^3 - 3\lambda k^2 - 4\lambda k^2 W^{-1} \right) + \frac{3}{2} k^4 \right] \left. \right] \exp (i\theta_0) + \ldots \quad \ldots (61)$$

The particular solution of third order eqs. (53)-(60) is given as

$$\eta_3 = \exp (\pm 2i\theta_0) \text{ terms } + \exp (\pm 3i\theta_0) \text{ terms } + \ldots , \quad \ldots (62)$$

$$\phi_3^{(1)} = \left[\frac{1}{k} \frac{\partial A}{\partial t_2} + \frac{\omega}{k^2} \frac{\partial A}{\partial x_2} + \left(V^{(1)} - \frac{\omega}{k} \right) \frac{\partial A}{\partial x_2} y + i \left(\frac{1}{k^2} - \frac{1}{k} y \right) \frac{\partial^2 A}{\partial x_1 \partial t_1} \right.$$

$$\left. + i \frac{\omega}{k^3} \frac{\partial^2 A}{\partial x_1^2} - i \frac{\omega}{k^2} \frac{\partial^2 A}{\partial x_1^2} y - \frac{i}{2} \left(V^{(1)} - \frac{\omega}{k} \right) \frac{\partial^2 A}{\partial x_1^2} y^2 \right.$$

$$+\frac{i}{2}k\left(V^{(1)}-\frac{\omega}{k}\right)(-6\lambda+k)A^2\bar{A}\Bigg]\exp(i\theta_0+ky)+\ldots, \quad \ldots (63)$$

$$\phi_3^{(2)}=\Bigg[-\frac{1}{k}\frac{\partial A}{\partial t_2}-\frac{\omega}{k^2}\frac{\partial A}{\partial x_2}+\left(V^{(2)}-\frac{\omega}{k}\right)\frac{\partial A}{\partial x_2}y-i\left(\frac{1}{k^2}+\frac{1}{k}y\right)\frac{\partial^2 A}{\partial x_1\,\partial t_1}$$

$$-i\frac{\omega}{k^3}\frac{\partial^2 A}{\partial x_1^2}-i\frac{\omega}{k^2}\frac{\partial^2 A}{\partial x_1^2}y+\frac{i}{2}\left(V^{(2)}-\frac{\omega}{k}\right)\frac{\partial^2 A}{\partial x_1^2}y^2$$

$$-\frac{i}{2}k\left(V^{(2)}-\frac{\omega}{k}\right)(6\lambda+k)A^2\bar{A}\Bigg]\exp(i\theta_0-ky)+\ldots, \quad \ldots (64)$$

$$\psi_3^{(1)}=\left[-iWE^{(1)}\frac{\partial A}{\partial x_2}y+C_\psi^{(1)}A^2\bar{A}-\frac{1}{2}WE^{(1)}\frac{\partial^2 A}{\partial x_1^2}y^2\right]\exp(i\theta_0+ky)+\ldots \quad \ldots (65)$$

and

$$\psi_3^{(2)}=\left[-iWE^{(2)}\frac{\partial A}{\partial x_2}y+C_\psi^{(2)}A^2\bar{A}+\frac{1}{2}WE^{(2)}\frac{\partial^2 A}{\partial x_1^2}y^2\right]\exp(i\theta_0-ky)+\ldots, \quad \ldots (66)$$

where

$$C_\psi^{(1)}=W/(e^{(2)}+e^{(1)})\left[-2\lambda k e^{(1)}E^{(1)}+\frac{1}{2}ke^{(2)}\left\{E^{(2)}(6\lambda+k)-E^{(1)}(6\lambda-k)\right\}\right] \quad \ldots (67)$$

and

$$C_\psi^{(2)}=W/(e^{(2)}+e^{(1)})\left[-2\lambda k e^{(1)}E^{(1)}-\frac{1}{2}ke^{(1)}\left\{E^{(2)}(6\lambda+k)-E^{(1)}(6\lambda-k)\right\}\right]. \quad \ldots (68)$$

These solutions of the third order problem yield the following solvability condition for η_3 from equation (61) (using $V^{(1)}=0$ & $V^{(2)}=V$) :

$$i\left(-D_\omega\frac{\partial A}{\partial t_2}+D_k\frac{\partial A}{\partial x_2}\right)+\frac{1}{2}D_{\omega\omega}\frac{\partial^2 A}{\partial t_1^2}-D_{\omega k}\frac{\partial^2 A}{\partial x_1\,\partial t_1}+\frac{1}{2}D_{kk}\frac{\partial^2 A}{\partial x_1^2}$$

$$=\mu D_V A+GA^2\bar{A}, \quad \ldots (69)$$

where the nonlinear coefficient is given by

$$G=-2\lambda\left\{\omega^2-\rho(\omega-kV)^2+\delta_2 k^2\right\}-2k\left\{\omega^2+\rho(\omega-kV)^2-\delta_0 k^2\right\}+1.5\,k^4, \quad \ldots (70)$$

where

$$\delta_2=-W\delta_0, \quad \ldots (71)$$

and

$$\lambda - \left[\omega^2 = \rho k^2 \left(V - \frac{\omega}{k} \right)^2 + k^2 \delta_2 \right] / (1 - \rho - 2k^2). \quad \ldots (72)$$

Let us now apply (52) and (69) in the neighbourhood of the minimum of the neutral curve, the critical point (k_c, V_c) for the instability. At the minimum of the neutral surface $(V = V_c, k = k_c)$, both first derivatives of D vanish, i.e., $D_\omega = 0$ and $D_k = 0$ at the critical point. With D_ω and D_k both equal to zero, eq. (52) disappears, and eq. (69), the sole governing equation for the amplitude, becomes

$$\frac{1}{2} D_{\omega\omega} \frac{\partial^2 A}{\partial t^2} - D_{\omega k} \frac{\partial^2 A}{\partial x_1 \partial t_1} + \frac{1}{2} D_{kk} \frac{\partial^2 A}{\partial x_1^2} = \mu D_V A + G A^2 \bar{A}, \quad \ldots (73)$$

where the various coefficients are evaluated at the critical point and G is the nonlinear interaction parameter, which can be either positive or negative. We may write (73) as

$$\left(\frac{\partial}{\partial t_1} + C \frac{\partial}{\partial x_1} \right)^2 A - \Omega^2 \frac{\partial^2 A}{\partial x_1^2} = \alpha A + \beta A^2 \bar{A}, \quad \ldots (74)$$

where

$$C = \frac{\rho V}{1 + \rho}, \quad \ldots (75)$$

$$\Omega^2 = \frac{[1 - \rho]^{1/2}}{1 + \rho}, \quad \ldots (76)$$

$$\alpha = 2\mu \frac{D_V}{D_{\omega\omega}}$$

$$= 2 ((1 - \rho)(1 + \rho)) \left(\frac{\rho}{1 + \rho} \right)^{1/2} [2(1 - \rho)^{1/2} + \delta_0] \left[\frac{V}{V_c} - 1 \right] / \varepsilon^2 \quad \ldots (77)$$

and

$$\beta = \frac{2G}{D_{\omega\omega}}$$

$$= \frac{1}{2} \frac{[1 - \rho]^{5/2}}{1 + \rho} \left[16 \left\{ \frac{(\rho - 1)}{(\rho + 1)} + \frac{1}{2[1 - \rho]^{1/2}} \frac{\delta_2 (1 + \rho) + \delta_0 (\rho - 1)}{1 + \rho} \right\}^2 - 5 \right]. \quad \ldots (78)$$

Eq. (74) can be expressed as a nonlinear Klein-Gordon equation

$$\frac{\partial^2 A}{\partial T^2} - \Omega^2 \frac{\partial^2 A}{\partial X^2} = \alpha A + \beta A^2 \bar{A}, \quad \ldots (79)$$

with the transformation

$$X = x_1 - ct_1, T = t_1.$$

It is interesting to note that the nonlinear interaction parameter β appearing in (79) exerts a destabilizing influence whenever

$$\frac{(\rho-1)}{(\rho+1)} + \frac{1}{2[1-\rho]^{1/2}} \frac{\delta_2(1+\rho)+\delta_0(\rho-1)}{1+\rho} > \left(\frac{5}{16}\right)^{1/2} \quad \ldots (80)$$

Thus, the nonlinear coefficient is destabilizing if

$$E^{(1)} E^{(2)} < \frac{2[1-\rho]^{1/2}(1+\rho)W^{-2}}{[(e^{(2)}+e^{(1)})(\rho-1)-(e^{(2)}-e^{(1)})(1+\rho)]} \left[\frac{(\rho-1)}{(\rho+1} - \left(\frac{5}{16}\right)^{1/2} \right] \quad \ldots (81)$$

For $e^{(2)} = e^{(1)}$, the coefficient β is negative when $0.28286 < \rho < 1$ and positive when $\rho < 0.28286$, leading to soft and hard excitations. Thus we have recovered the corresponding result of Weissman for the pure hydrodynamic case by setting $e^{(1)} = e^{(2)}$.

We shall now examine various special cases of physical interest when the spatial modulations of the amplitude are assumed to be negligible. For simplicity, we assume that A is purely real, which corresponds to fixing the phase of the complex amplitude. Therefore, eq. (79) is reduced to

$$\frac{\partial^2 A}{\partial T^2} = \alpha A + \beta A^3. \quad \ldots (82)$$

The solutions of eq. (82) can be found in terms of Jacobi elliptic functions. The character of the solutions depends sensitively on the signs of the coefficients α and β.

The first integral of (82) is

$$C = \left(\frac{dA}{dT}\right)^2 + \Phi(A) = \text{constant}, \quad \ldots (83)$$

where $\Phi(A) = -\alpha A^2 - (\beta/2)A^4$. The function $\Phi(A)$ can be regarded as potential of (82). The solutions are classified into four cases :

Case (i) $\alpha < 0, \beta < 0$

Since both the linear term and the nonlinear term are stabilizing, all solutions are bounded. The phase portrait is shown in Fig. 2. We note from the figure that the nonlinearity generates stabilizing influence leading to bounded solutions. For motions in the vicinity of the origin, the A^4 term is negligible and what's left is the equation of a family of ellipses, $C = \left(\frac{dA}{dT}\right)^2 - \alpha A^2$. For larger motions, the A^4 term becomes important, and the curves are essentially distorted ellipses. The phase plane analysis reveals that the trajectories are closed curves representing the Jacobi elliptic *cn* functions.

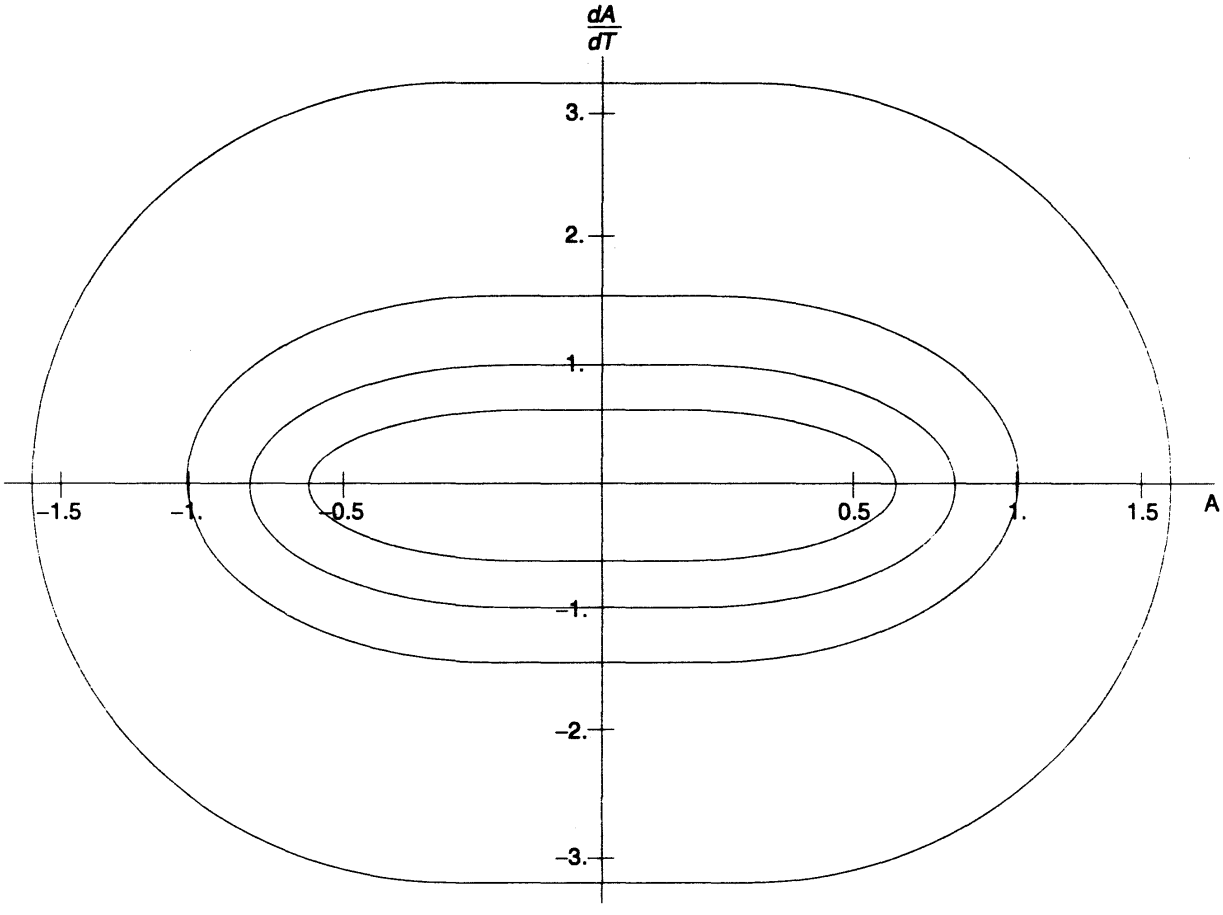

FIG. 2. [$\alpha < 0, \beta < 0$]

Case (ii) $\alpha > 0, \beta > 0.$

Since both the linear and the nonlinear terms are destabilizing, the instability amplitude $A(T)$ increases super-exponentially, leading to "explosive" instability. The solution breaks down after a finite time. In fact, this kind of breakdown may not occur in practice because the dissipative mechanism such as viscosity is likely to check the growth rate. The phase protrait is shown in Fig. 3. We note that there exist no bounded solutions. We also note that there exist only one equilibrium state in the phase protrait at the origin $A = \frac{dA}{dT} = 0.$

Case (iii) $\alpha < 0, \beta > 0.$

The plot of integral curves (83) is shown in Fig. 4. We notice that three singularities determine the essential structure of the entire phase plane. The curves through the singular points $A = \pm\left(\frac{|\alpha|}{\beta}\right)^{1/2}, \frac{dA}{dT} = 0,$ are of special importance because they separate qualitatively different kinds of motions. Such a trajectory is called "separatrix". Also we note that near $A = \frac{dA}{dT} = 0,$ the curves are essentially elliptical, but as the amplitude increases, the ellipses begin to distort and become more pointed at the ends. The degree of distortion is a measure of the effects of nonlinearity. Once outside the separatrix, the elliptical shape is totally lost and nonlinear effects predominate. Thus,

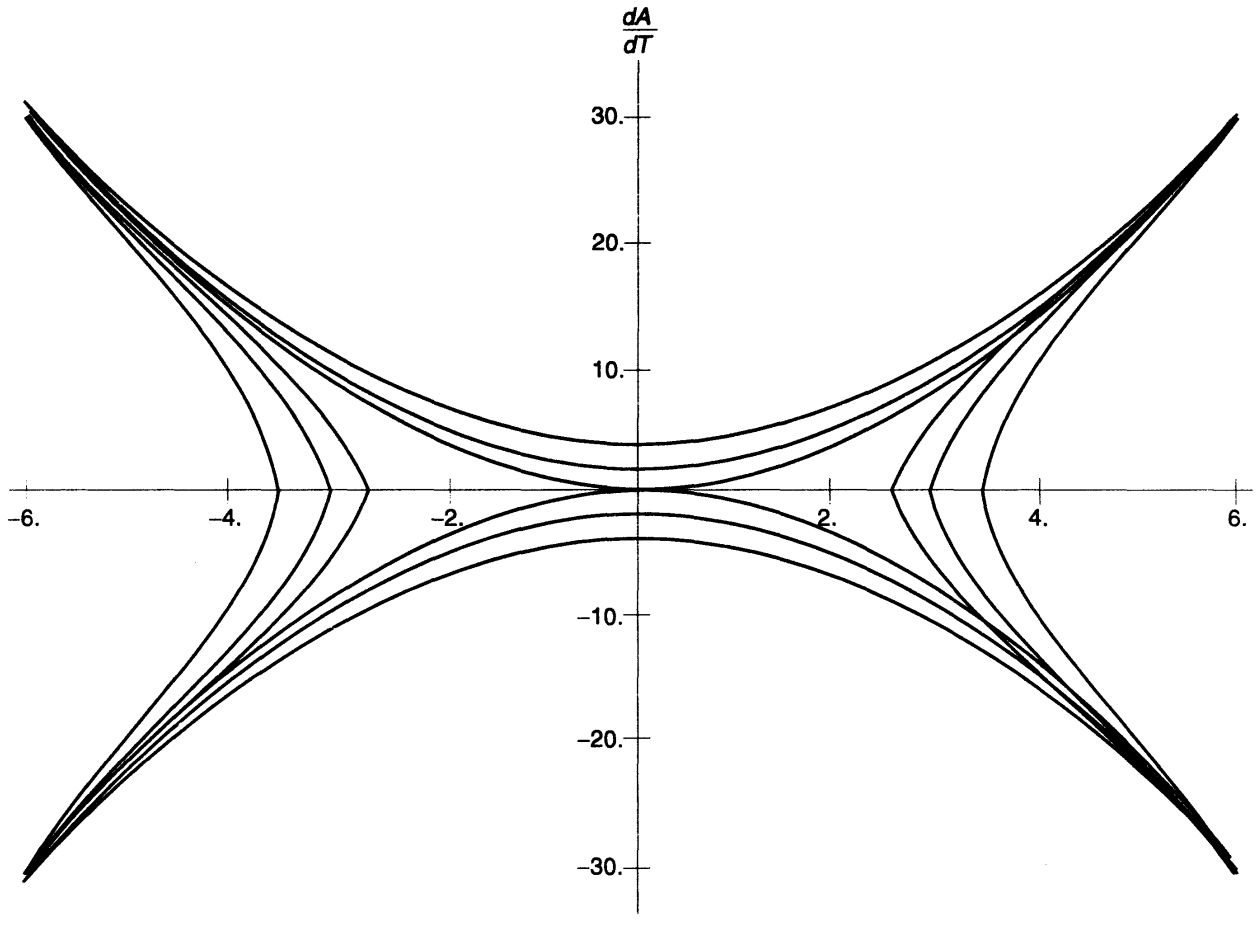

FIG. 3. [$\alpha > 0, \beta > 0$]

when $\alpha < 0, \beta > 0$, the two terms on the right-hand side of eq. (82) compete with each other giving the saturation amplitude :

$$A_{eq} = \left(\frac{|\alpha|}{\beta} \right)^{1/2} \qquad \ldots (84)$$

In this case, A_{eq} is a threshold value in the sense that if the initial amplitude $|A_0| < A_{eq}$ then the solution is bounded as T tends to ∞, is shown by closed trajectories near the origin of the phase plane and is given in terms of Jacobi elliptic *sn* function and if $|A_0| > A_{eq}$, then the solution breaks down after a finite time. This leads to an instability. Such an instability is termed as 'sub-critical' since the instability sets in for the finite amplitude even when all the linear perturbations are stable.

Case (iv) $\alpha > 0, \beta < 0$.

The phase portrait is sketched in Fig. 5. We observe that there exist three equilibrium states in the phase plot : one at the origin and two at the points $A = \pm A_{eq} = \pm \left(\frac{\alpha}{|\beta|} \right)^{1/2}$. These states correspond to the bifurcation of equilibrium, and are not dynamically connected with one another, so that neither of them can be reached by the others through the evolution of instability. For

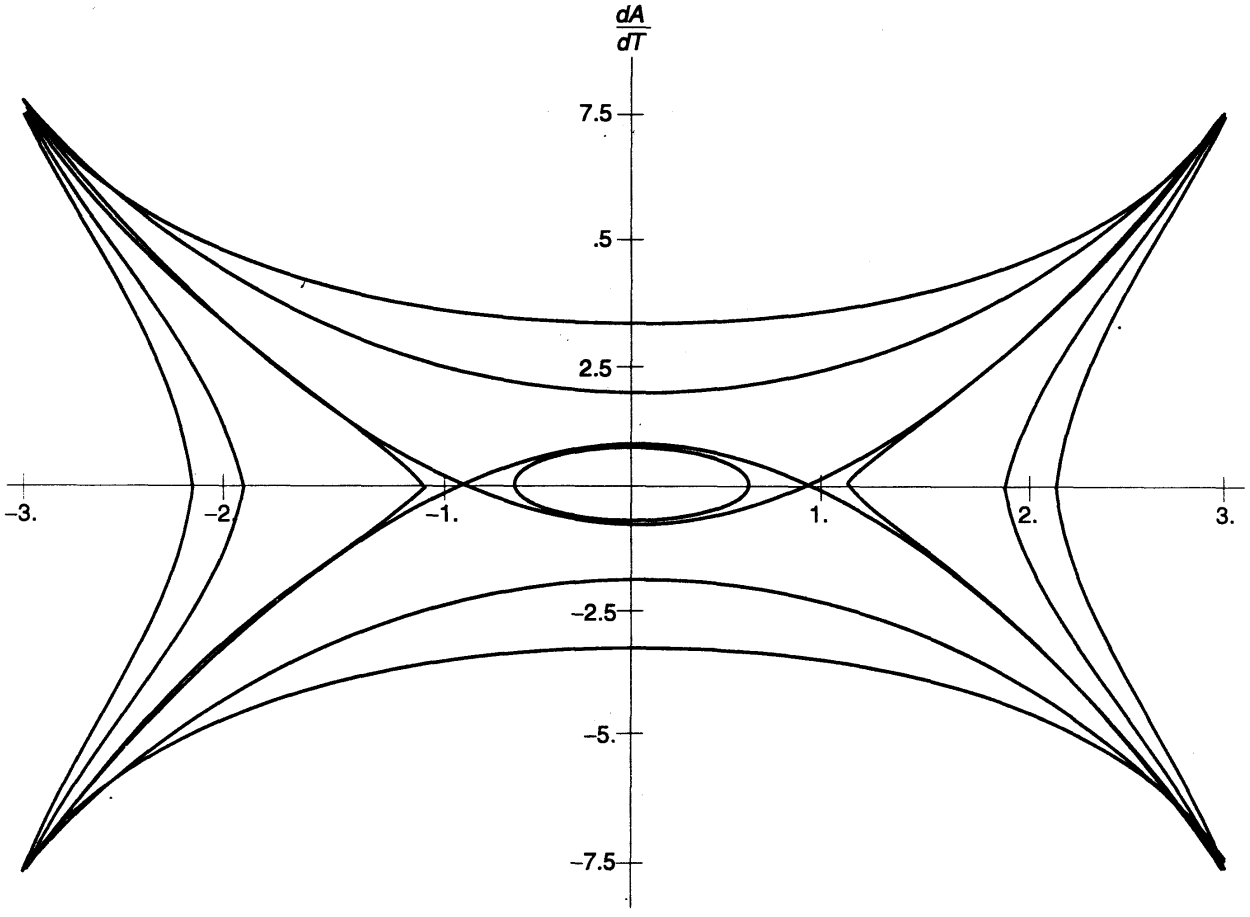

FIG. 4. [$\alpha < 0, \beta > 0$]

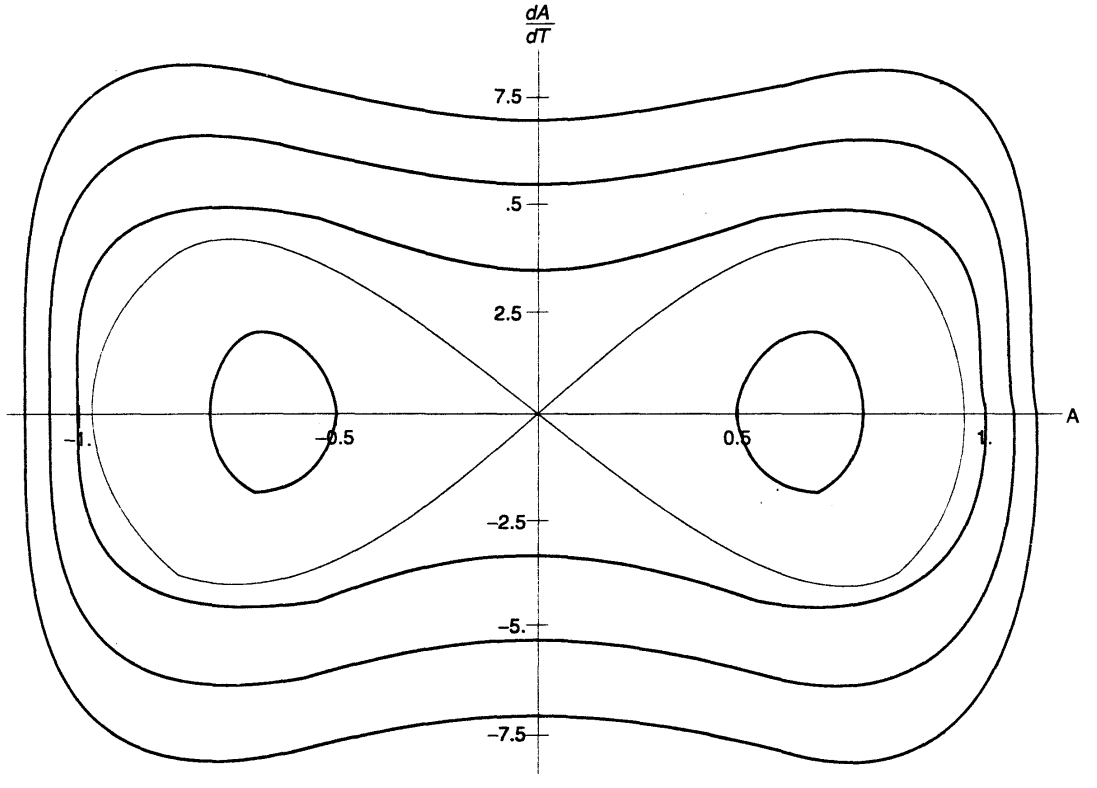

FIG. 5. [$\alpha > 0, \beta < 0$]

$C = 0$, the amplitude $A(T)$ grows exponentially, passes through the maximum at $A^2_{\max} = -2\frac{\alpha}{\beta}$, and finally decays exponentially to zero. Thus in this case, the amplitude is said to 'equilibrate'. Eq. (82) is linearly unstable but the nonlinear term is stabilizing, yielding bounded oscillating solutions of two different types.

1. Inside the separatrix ($\varphi(A) < 0$), near the stable equilibrium points $\left(A = \pm A_{eq}, \frac{dA}{dT} = 0 \right)$, the solutions do not change sign and are given in terms of the Jacobi elliptic dn function.

2. Outside the separatrix, on the trajectories where A does change sign, the solutions have the form of the Jacobi elliptic cn functions as in case (*i*).

3. On the separatrix, the solution is the hyperbolic secant.

4. CONCLUSIONS

The electrohydrodynamic stability of a horizontal interface separating two dielectric streaming fluids stressed by a normal electric field has been studied. The method of multiple scale perturbations has been used in order to obtain linear and successive nonlinear partial differential equations of various orders. The problem can be solved for any order with the knowledge of the solutions of the previous orders.

From the linearized problem, we have shown that both the streaming velocity and the perpendicular fields are strictly destabilizing. For $k = k_c$ and $V = V_c$ (the critical point), a generalized formulation of the evolution equation governing the amplitude is developed, which leads to the nonlinear Klein-Gordon equation. The stability conditions of the latter equation are obtained, and the general solution of that equation, which is the Jacobi elliptic function, is discussed. We find that the nonlinear effects can be stabilizing or destabilizing depending on both the density ratio and the dielectric constants.

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APPENDIX

We define the various operators as

$$D_{ij} = \frac{\partial^j}{\partial x_i^j}, D_j = \frac{\partial^j}{\partial y_j}, D_{i,p} = \frac{\partial^2}{\partial x_i \partial p}, \quad \nabla_0^2 = D_{02} + D_2,$$

$$N_{i,p} = \frac{\partial^2}{\partial t_i \, \partial p}, Q_{i,p} = \frac{\partial^3}{\partial t_i \, \partial p^2}, \Delta_{ij} = \frac{\partial^2}{\partial x_i \, \partial x_j}, \Delta_{i,p} = \frac{\partial}{\partial p_i},$$

$$M_{i,j} = \Delta_{i,t} + V^{(j)} \Delta_{i,x}, R_{i,j} = N_{i,y} + V^{(j)} D_{i,y},$$

$$L_0(a, b) = \Delta_{0,t} a - D_1 b + V^{(j)} \Delta_{0,x} a,$$

$$L_1(a, b, c, d) = (1 - \rho)a + M_{0,1} b - \rho M_{0,2} c - D_{02} a + [\![eED_1 d]\!],$$

$$S(a, b) = \eta_2 D_{0,y} a + 0.5 \eta_1^2 D_2 \Delta_{0,x} a + \eta_1 D_{0,y} b + \eta_1 D_{1,y} a$$

$$+ \Delta_{1,x} b + \Delta_{2,x} a. \qquad \ldots (A.1)$$

The equations of the first-order problem $o(\varepsilon)$ are (j = 1, 2) :

$$\nabla_0^2 \phi_1^{(j)} = 0, \nabla_0^2 \psi_1^{(j)} = 0, \qquad \ldots (A.2)$$

with the boundary conditions at y = 0 :

$$L_0(\eta_1, \phi_1^{(j)}) = 0, \qquad \ldots (A.3)$$

$$[\![\Delta_0, x^{\psi_1}]\!] - \Delta_{0,x_1}^{\eta} [E] = 0, \qquad \ldots (A.4)$$

$$[\![eD_1 \psi_1]\!] = 0, \qquad \ldots (A.5)$$

$$L_1(\eta_1, \phi_1^{(1)}, \phi_1^{(2)}, \psi_1) = 0. \qquad \ldots (A.6)$$

The equations of the second-order problem $o(\varepsilon^2)$ are (j = 1, 2) :

$$\nabla_0^2 \phi_2^{(j)} = -2\Delta_{01}\, \phi_1^{(j)},\ \nabla_0^2 \psi_2^{(j)} = -2\Delta_{01}\, \psi_1^{(j)}\ , \qquad \text{... (A.7)}$$

with the boundary conditions at y = 0 :

$$L_0(\eta_2, \phi_2^{(j)}) = -M_{1,j}\,\eta_1 + \eta_1 D_2 \phi_1^{(j)} - (\Delta_{0,x}\eta_1)\,(\Delta_{0,x}\,\phi_1^{(j)}), \text{... (A.8)}$$

$$\left[\!\left[\ \Delta_{0,x}\psi_2\ \right]\!\right] - \Delta_{0,x}\,\eta_2\,[\,E\ \ right] = -\left[\!\left[\ \Delta_{1,x}\,\psi_1\ \right]\!\right] + \Delta_{1,x}\,\eta_1\,[\,E\,] - \eta_1\left[\!\left[\ D_{0,y}p_1\ \right]\!\right] - (\Delta_{0,x}\eta_1)\left[\!\left[\ D_1\,\psi_1\ \right]\!\right], \qquad \text{... (A.9)}$$

$$\left[\!\left[\ eD_1\,\psi_2\ \right]\!\right] = (\Delta_{0,x}\,\eta_1)\left[\!\left[\ e\Delta_{0,x}\,\psi_1\ \right]\!\right] - \eta_1\left[\!\left[\ eD_2\,\psi_1\ \right]\!\right], \text{... (A.10)}$$

$$L_1\,(\eta_2, \phi_2^{(1)}, \phi_2^{(2)}, \psi_2) = -\eta_1\left[\ R_{0,1}\,\phi_1^{(1)} - \rho R_{0,2}\,\phi_1^{(2)}\ \right] - M_{1,1}\,\phi_1^{(1)}$$

$$+\ \rho M_{1,2}\,\phi_1^{(2)} + 2\Delta_{01}\,\eta_1 - 0.5\left[\left(\Delta_{0,x}\phi_1^{(1)}\right)^2 - \rho\left(\Delta_{0x}\phi_1^{(2)}\right)^2\right.$$

$$\left.+\ (D_1\,\phi_1^{(1)})^2 - \rho\,(D_1\phi_1^{(2)})^2\ \right] - 0.5\left[\!\left[\ e\,(\Delta_{0,x}\psi_1)^2\ \right]\!\right] - \eta_1\left[\!\left[\ eED_2\,\psi_1\ \right]\!\right]$$

$$+\ 0.5\left[\!\left[\ e(D_1\,\psi_1)^2\ \right]\!\right] + 2(\Delta_{0,x}\,\eta_1)\left[\!\left[\ eE\Delta_{0,x}\,\psi_1\ \right]\!\right] - (\Delta_{0,x}\,\eta_1)^2\left[\!\left[\ eE^2\ \right]\!\right] \qquad \text{... (A.11)}$$

The equations of the third-order problem $o(\varepsilon^3)$ are (j = 1, 2) :

$$\nabla_0^2 \phi_3^{(j)} = -2\Delta_{01}\,\phi_2^{(j)} - 2\Delta_{02}\,\phi_1^{(j)} - D_{12}\,\phi_1^{(j)}$$

and

$$\nabla_0^2 \psi_3^{(j)} = -2\Delta_{01}\,\psi_2^{(j)} - 2\Delta_{0,2}\,\psi_1^{(j)} - D_{12}\,\psi_1^{(j)}\ , \qquad \text{... (A.12)}$$

with the boundary conditions at y = 0 :

$$L_0\,(\eta_3, \phi_3^{(j)}) = -M_{2,j}\,\eta_1 - M_{i,j}\,\eta_2 + \eta_1\,D_2\,\phi_2^{(j)} + \eta_2\,D_2\,\phi_1^{(j)} + 0.5\,\eta_1^2\,D_3\,\phi_1^{(j)}$$

$$-\ (\Delta_{0,x}\,\eta_1)\left[\ \Delta_{0,x}\,\phi_2^{(j)} + \Delta_{1,x}\,\phi_1^{(j)} + \eta_1\,D_{0,y}\phi_1^{(j)}\ \right]$$

$$-\ (\Delta_{0,x}\,\phi_1^{(j)})\left[\ \Delta_{0,x}\eta_2 + \Delta_{1,x}\,\eta_1\ \right] \qquad \text{... (A.13)}$$

$$\left[\!\left[\ \Delta_{0,x}\,\psi_3\ \right]\!\right] - \Delta_{0,x}\,\eta_3\,[\,E\,] = -\left[\!\left[\ \Delta_{2,x}\,\psi_1\ \right]\!\right] - \left[\!\left[\ \Delta_{1,x}\,\psi_2\ \right]\!\right] + \Delta_{1,x}\,\eta_2\,[\,E\,]$$

$$+\ \Delta_{2,x}\,\eta_1\,[\,E\,] - \eta_1\,(\Delta_{0,x}\,\eta_1)\left[\!\left[\ D_2\,\psi_i\ \right]\!\right] - (\Delta_{0,x}\,\eta_1)\left[\!\left[\ D_1\,\psi_2\ \right]\!\right] - (\Delta_{0,x}\,\eta_2)\left[\!\left[\ D_1\,\psi_1\ \right]\!\right]$$

$$-(\Delta_{1,x}\eta_1)\left[\!\left[\ D_1\psi_1\ \right]\!\right]-\eta_2\left[\!\left[\ D_{0,y}\psi_1\ \right]\!\right]-0.5\,\eta_1^2\left[\!\left[\ D_2\Delta_{0,x}\psi_1\ \right]\!\right]$$

$$-\eta_1\left[\!\left[\ D_{0,y}\psi_2\ \right]\!\right]-\eta_1\left[\!\left[\ D_{1,y}\psi_1\ \right]\right] \qquad \ldots \text{(A.14)}$$

$$\left[\!\left[\ eD_1\psi_3\ \right]\!\right]=\eta_1(\Delta_{0,x}\eta_1)\left[\!\left[\ eD_{0,y}\psi_1\ \right]\!\right]+(\Delta_{0,x}\eta_1)\left[\!\left[\ e\Delta_{0,x}\psi_2\ \right]\!\right]$$

$$+(\Delta_{0,x}\eta_1)\left[\!\left[\ e\Delta_{1,x}\psi_1\ \right]\!\right]+(\Delta_{0,x}\eta_2+\Delta_{1,x}\eta_1)\left[\!\left[\ e\Delta_{0,x}\psi_1\ \right]\right]$$

$$-\eta_2\left[\!\left[\ eD_2\psi_1\ \right]\!\right]-0.5\,\eta_1^2\left[\!\left[\ eD_3\psi_1\ \right]\!\right]-\eta_1\left[\!\left[\ eD_2\psi_2\ \right]\!\right], \qquad \ldots \text{(A.15)}$$

$$L_1(\eta_3,\phi_3^{(1)},\phi_3^{(2)},\psi_3)=-V^{(1)}S(\phi_1^{(1)},\phi_2^{(1)})+\rho V^{(2)}S(\phi_1^{(2)},\phi_2^{(2)})$$

$$-\Delta_{2,t}\phi_1^{(1)}-\Delta_{1,t}\phi_2^{(1)}+\rho\Delta_{2,t}\phi_1^{(2)}+\rho\Delta_{1,t}\phi_2^{(2)}+2\Delta_{02}\eta_1+2\Delta_{01}\eta_2+D_{12}\eta_1$$

$$-(\Delta_{0,x}\phi_1^{(1)})\left[(\Delta_{0,x}\phi_2^{(1)})+(\Delta_{1,x}\phi_1^{(1)})\right]-\eta_1(\Delta_{0,x}\phi_1^{(1)})(D_{0,y}\phi_1^{(1)})$$

$$+\rho(\Delta_{0,x}\phi_1^{(2)})\left[(\Delta_{0,x}\phi_2^{(2)})+(\Delta_{1,x}\phi_1^{(2)})\right]+\rho\eta_1(\Delta_{0,x}\phi_1^{(2)})(D_{0,y}\phi_1^{(2)})$$

$$-(D_1\phi_1^{(1)})\left[D_1\phi_2^{(1)}+\eta_1 D_2\phi_1^{(1)}\right]+\rho(D_1\phi_1^{(2)})\left[D_1\phi_2^{(2)}+\eta_1 D_2\phi_1^{(2)}\right]$$

$$-\eta_2 N_{0,y}\phi_1^{(1)}-\eta_1 N_{0,y}\phi_2^{(1)}-0.5\,\eta_1^2 Q_{0,y}\phi_1^{(1)}+\rho\eta_2 N_{0,y}\phi_1^{(2)}+\rho\eta_1 N_{0,y}\phi_2^{(2)}$$

$$+0.5\,\rho\eta_1^2 Q_{0,y}\phi_1^{(2)}-\eta_1 N_{1,y}\phi_1^{(1)}+\rho\eta_1 N_{1,y}\phi_1^{(2)}-1.5\,(D_{02}\eta_1)(\Delta_{0,x}\eta_1)^2$$

$$-\eta_1\left[\!\left[\ e\Delta_{0,x}\psi_1 D_{0,y}\psi_1\ \right]\!\right]-\left[\!\left[\ e\Delta_{0,x}\psi_1\Delta_{0,x}\psi_2\ \right]\!\right]-\left[\!\left[\ e\Delta_{0,x}\psi_1\Delta_{1,x}\psi_1\ \right]\!\right]$$

$$-\eta_2\left[\!\left[\ eED_2\psi_1\ \right]\!\right]-0.5\,\eta_1^2\left[\!\left[\ eED_3\psi_1\ \right]\!\right]-\eta_1\left[\!\left[\ eED_2\psi_2\ \right]\!\right]$$

$$+\eta_1\left[\!\left[\ eD_1\psi_1 D_2\psi_1\ \right]\!\right]+\left[\!\left[\ eE\Delta_{0,x}\psi_2\ \right]\!\right]+\left[\!\left[\ eE\Delta_{1,x}\psi_1\ \right]\!\right]$$

$$\left[\!\left[\ eD_1\psi_1 D_1\psi_2\ \right]\!\right]+2(\Delta_{0,x}\eta_1)\left[\!\left[\ \eta_1\left[\!\left[\ eED_{0,y}\psi_1\ \right]\!\right]\right.\right.$$

$$\left.\left.-\left[\!\left[\ e\Delta_{0,x}\psi_1 D_1\psi_1\ \right]\!\right]-(\Delta_{0,x}\eta_2+\Delta_{1,x}\eta_1)\left[\!\left[\ eE^2\ \right]\!\right]+(\Delta_{0,x}\eta_1)\left[\!\left[\ eED_1\psi_1\ \right]\!\right]\right]\right]$$

$$+2(\Delta_{0,x}\eta_2)\left[\!\left[\ eE\Delta_{0,x}\psi_1\ \right]\right]+2(\Delta_{1,x}\eta_1)\left[\!\left[\ eE\Delta_{0,x}\psi_1\ \right]\right]. \qquad \ldots \text{(A.16)}$$