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STRUCTURE ON A SLANT SUBMANIFOLD OF A CONTACT MANIFOLD

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In this paper, we study the possibility of obtaining an induced contact metric structure on a slant submanifold of a contact metric manifold. We also give a characterization theorem for three-dimensional slant submanifolds.

Key Words : Contact Manifold; Sasakian Manifold; Slant Submanifold

INTRODUCTION

Slant immersions in complex geometry were defined by Chen as a natural generalization of both holomorphic immersions and totally real immersions[3]. In a recent paper[4], A. Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold. We have studied and characterized slant submanifolds in K-contact and Sasakian manifolds and we have given several examples of such immersions in [2].

The purpose of the present paper is to study the possibility of defining an induced contact metric structure on a slant submanifold. The tools used will allow to give a characterization result when the submanifold has dimension 3.

In Section 1 we review basic formulas and definitions for almost contact metric manifolds and their submanifolds, which we shall use later. We also review some definitions and properties given in [2, 4]. In Section 2, we prove the main theorem of this paper. This result states that there are no non-invariant slant immersions from a three-dimensional contact metric manifold in another contact metric manifold, with compatible structure vector fields.

1. PRELIMINARIES

Let $(\tilde{M}, g)$ be an odd-dimensional Riemannian manifold and denote by $T\tilde{M}$ the Lie algebra of vector fields in $\tilde{M}$. Then $\tilde{M}$ is said to be an almost contact metric manifold[1] if there exists on $\tilde{M}$ a tensor ϕ of type (1, 1) and a global vector field ξ (structure vector field) such that, if η is the dual 1-form of ξ, then

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$$\phi^2 X = -X + \eta(X)\xi,\ g(X, \xi) = \eta(X),$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y),$$

for any $X, Y \in T\tilde{M}$. In this case,

$$g(\phi X, Y) + g(X, \phi Y) = 0,$$

for any $X, Y \in T\tilde{M}$. Let Φ denote the 2-form in $\tilde{M}$ given by $\Phi(X, Y) = g(X, \phi Y)$ for all $X, Y \in T\tilde{M}$. The 2-form Φ is called the fundamental 2-form in M and the manifold is said to be a contact metric manifold if $\Phi = d\eta$. If ξ is a Killing vector field with respect to g, the contact metric structure is called a K-contact structure. It is easy to prove that a contact metric manifold is K-contact if and only if $\tilde{\nabla}_X \xi = -\phi X$, for any $X \in T\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection of $\tilde{M}$.

The almost contact structure of $\tilde{M}$ is said to be normal if $[\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A Sasakian manifold is a normal contact metric manifold. Every Sasakian manifold is a K-contact manifold. It is easy to show that an almost contact metric manifold is a Sasakian manifold if and only if

$$(\tilde{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any $X, Y \in T\tilde{M}$.

Now, let M be a submanifold immersed in $\tilde{M}$. We also denote by g the induced metric on M. Let TM be the Lie algebra of vector fields in M and $T^{\perp}M$ the set of all vector fields normal to M. Denote by ∇ the Levi-Civita connection of M.

For any $X \in TM$, we write

$$\phi X = TX + NX,$$

where TX is the tangential components of ϕX and NX is the normal component of ϕX. Then T is an endomorphism of the tangent bundle and N is a normal-bundle valued 1-form on the tangent bundle.

The submanifold M is said to be invariant if N is identically zero, that is, $\phi X \in TM$, for any $X \in TM$. On the other hand, M is said to be an anti-invariant submanifold if T is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

From now on, we suppose that the structure vector field is tangent to M. Hence, if we denote by $\mathcal{D}$ the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = \mathcal{D} \oplus \langle \xi \rangle$.

For each nonzero vector X tangent to M at x, such that X is not proportional to ξ_x, we denote by $\theta(X)$ the Wiritinger angle of X, that is, the angle between ϕX and $T_x M$.

Hence, according Lotta's definition, M is *slant* if the Writinger angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_x M - \langle \xi_x \rangle$ (see [4]). The Wiritinger angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle θ equal to 0 and $\pi/2$ respectively. A slant immersion which is not invariant nor anti-invariant is called a proper slant immersion.

In [2], we have shown the following results:

Theorem 1.1 — *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that :*

$$T^2 = -\lambda I + \lambda \eta \otimes \xi. \qquad \text{... (1.1)}$$

Furthermore, in such a case, if θ is the slant angle of M, $\lambda = \cos^2 \theta$.

Corollary 1.2 — Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$, with slant angle θ. Then, for any $X, Y \in TM$, we have :

$$g(TX, TY) = \cos^2 \theta(g(X, Y) - \eta(X)\eta(Y)), \qquad \text{... (1.2)}$$

and

$$g(NX, NY) = \sin^2 \theta(g(X, Y) - \eta(X)\eta(Y)). \qquad \text{... (1.3)}$$

Set $Q = T^2$. We also need the following results, given in [2]:

Proposition 1.3 — *Let M be a slant submanifold of a K-contact manifold $\tilde{M}$. Denote by θ the slant angle of M. Then, we have*

$$(\nabla_X Q)Y = \cos^2 \theta(g(X, TY)\, \xi - \eta(Y)TX),$$

for any $X, Y \in TM$.

Proposition 1.4 — *Let M be a manifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. If there exists a function λ such that $(\nabla_X T)Y = \lambda(g(X, Y)\, \xi - \eta(Y)X)$ for any $X, Y \in TM$, then $(\nabla_X Q)Y = \lambda(g(X, TY)\xi - \eta(Y)TX)$ for any $X, Y \in TM$.*

2. Structure on a Slant Submanifold

Let M be a submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in TM$. It is well known that, if M is an invariant submanifold, then the structure of $\tilde{M}$ induces by a natural way a Sasakian structure over M. In this case, the submanifold is usually called a Sasakian submanifold. The purpose of this paper is to study if we can obtain an induced Sasakian structure on a non-invariant slant submanifold.

This problem is suggested by the similar situation on slant submanifolds of a Kaehlerian manifold. In [3], Chen gives the notion of a Kaehlerian slant submanifold of an almost Hermitian manifold $(\tilde{M}, J, g)$ as a proper slant submanifold such that the tangential component P of the almost complex structure J is parallel, that is, $\nabla P = 0$. It is easy to show that a Kaehlerian slant submanifold is a Kaehlerian manifold with respect to the induced metric and with the almost complex structure given by $\bar{J} = (\sec \theta)P$, where θ denotes the slant angle.

In the almost contact case, we can first give the following lemma:

Lemma 2.1 — Let M be a non-anti-invariant slant submanifold of an almost contact metric manifold $\tilde{M}$. The, M is an almost contact metric manifold with respect to the induced metric, with structure vector field ξ and with the almost contact structure given by $\bar{\phi} = (\sec \theta)T$, where θ denotes the slant angle of M.

PROOF : By virtue of (1.1) and (1.2), it is easy to show that $\bar{\phi}^2 X = -X + \eta(X)\xi$ and $g(\bar{\phi}X, \bar{\phi}Y) = g(X, Y) - \eta(X)\, \eta(Y)$, for any $X, Y \in TM$. □

In particular, if θ = 0, then the induced structure on the invariant submanifold M is the usual one.

Therefore, we want to find an appropriate condition on ∇T in order for it to be possible to induce a Sasakian structure on M.

In contact geometry, the similar notion to Kaehlerian slant submanifolds is given by proper θ-slant submanifolds satisfying

$$(\nabla_X T)Y = \cos^2 \theta(g(X, Y)\xi - \eta(Y)X), \quad \text{... (2.1)}$$

for any tangent fields X, Y, as we have shown in [2]. But, in this case, the almost contact metric structure given by $\overline{\phi}$ is not a Sasakian structure, since, from (2.1), it is easy to see that $(\nabla_X \overline{\phi})Y = \cos \theta(g(X, Y)\xi - \eta(Y)X)$, for any $X, Y \in TM$. However, we can modify the condition (2.1) to obtain a Sasakian structure on M. It can be proved that if

$$(\nabla_X T)Y = \cos \theta(g(X, Y)\xi - \eta(Y)X), \quad \text{... (2.2)}$$

for any $X, Y \in TM$, then M has an induced Sasakian structure given by $\overline{\phi}$. Nevertheless, we have the following result:

Proposition 2.2 — *There are no proper slant submanifolds M of a K-contact manifold satisfying eq. (2.2).*

PROOF : It follows directly from Proposition 1.4, Proposition 1.3 and (2.2). □

In fact, it is easy to see that, if M is a proper slant submanifold, then the structure induced by $\overline{\phi}$ is not a contact metric structure, because

$$\overline{\Phi}(X, Y) = g(X, \overline{\phi}Y) = \sec \theta g(X, TY) = \sec \theta d\eta(X, Y),$$

for any $X, Y \in TM$, since $\tilde{M}$ is a Sasakian manifold, and so, a contact metric manifold[4].

However, we can now wonder if it is possible to induce a Sasakian structure from another way, by choosing the appropriate conditions. We have the following intrinsic characterization of slant immersions in K-contact manifolds:

Theorem 2.3 — [4] *Let M be a submanifold of a K-contact manifold $\tilde{M}$, such that $\xi \in TM$. Let $\theta \in [0, \pi/2]$. The following statements are equivalent :*

(*i*) *M is slant in $\tilde{M}$, with slant angle θ.*

(*ii*) *For any $x \in M$, the sectional curvature of any 2-plane of T_xM containing ξ_x equals $\cos^2 \theta$.*

Then, we can state the following corollary:

Corollary 2.4 — Let M be a slant submanifold of a K-contact manifold $\tilde{M}$. Then, M is K-contact if and only if M is an invariant sub manifold.

PROOF : The direct implication follows directly from Theorem 2.3. The converse is a well-known result. □

Hence, it is not possible to have neither a Sasakian nor a K-contact induced structure on a non-invariant slant submanifold of a Sasakian manifold.

Nevertheless, we can still wonder if it would be possible to induce a contact metric structure. We are now going to use a different method.

Let

$$\varphi : (M, \overline{\phi}, \xi, \overline{\eta}, \overline{g}) \to (\tilde{M}, \phi, \xi, \eta, g)$$

be an immersion between two almost contact metric manifolds. Suppose this immersion to be isometric, i.e, $\bar{g} = \varphi^* g$, and such that $\varphi_{*x}\,\xi_x = \xi_{\varphi(x)}$ for any $x \in M$. In particular, it means that $\xi \in TM$.

Let $\bar{\Phi}$ and Φ be the fundamental 2-forms of M and $\tilde{M}$ respectively. We consider on M the 2-form $\varphi^*\Phi$ given by

$$\varphi^*\Phi(X, Y) = \Phi(\varphi_* X, \varphi_* Y) = g(\varphi_* X, \phi\varphi_x Y),$$

for any $X, Y \in \chi(M)$. From now on, we are identifying X and $\varphi_* X$ for any $X \in \chi(M)$.

If dim M = 3 and φ is a slant immersion, with slant angle θ, then we can find the following relation between $\bar{\Phi}$ and $\varphi^*\Phi$.

Proposition 2.5 — *In the above conditions,* $\varphi^*\Phi = \pm(\cos\theta)\,\bar{\Phi}$.

PROOF : If the immersion is anti-invariant, then the result is obvious since $\varphi^*\Phi = 0$. We can then suppose that φ is a non-anti-invariant slant immersion.

Let e_1 be a unit local field, tangent to M and perpendicular to ξ. If we put $e_2 = (\sec\theta)Te_1$, by virtue of Corollary 1.2, we know that $\{e_1, e_2, \xi\}$ is a local orthonormal frame of TM. Then, it is clear that $\bar{\phi}\, e_1 = g(\bar{\phi} e_1, e_2)e_2$, and so $g(\bar{\phi} e_1, \bar{\phi}\, e_1) = g^2(\bar{\phi} e_1, e_2)$. Now then, it is easy to see that $g(\bar{\phi} e_1, \bar{\phi} e_1) = 1$, from which we have:

$$\bar{\phi} e_1 = \pm e_2,\ \bar{\phi} e_2 = \mp e_1. \qquad \text{... (2.3)}$$

Let $X = X^1 e_1 + X^2 e_2 + \eta(X)\xi$, $Y = Y^1 e_1 + Y^2 e_2 + \eta(Y)\xi$ be two local tangent fields. Then, from (2.3), it results:

$$\bar{\Phi}(X, Y) = g(X, \bar{\phi} Y) = \mp X^1 Y^2 \pm X^2 Y^1. \qquad \text{... (2.4)}$$

On the other hand,

$$\varphi^*\Phi(X, Y) = \Phi(\varphi_* X, \varphi_* Y) = g(X, \phi Y) =$$

$$= g(X, TY) = -\cos\theta\, X^1 Y^2 + \cos\theta X^2 Y^1, \qquad \text{... (2.5)}$$

since $TY = -\cos\theta Y^2 e_1 + \cos\theta Y^1 e_2$. Hence, the result follows from eqs. (2.4) and (2.5). □

Now, by using Proposition 2.5, we obtain the main result of this paper:

Theorem 2.6 — *There are no non-invariant slant immersions* $\varphi : M \to \tilde{M}$ *from a contact metric manifold* $(M, \bar{\phi}, \xi, \bar{\eta}, \bar{g})$, *with dim* M = 3, *in another contact metric manifold* $(\tilde{M}, \phi, \xi, \eta, g)$ *such that* $\varphi_*\xi = \xi$.

PROOF : Suppose that there is a non-invariant slant immersion $\varphi : M \to \tilde{M}$ in the above conditions and denote by θ the slant angle of this immersion. Then, since φ is an isometric immersion and $\varphi_*\xi = \xi$, we have $\varphi^*\eta = \bar{\eta}$, and so

$$d(\varphi^*\eta) = d\bar{\eta} = \bar{\Phi}, \qquad \text{... (2.6)}$$

because M is a contact metric manifold. Now, from Proposition 2.5, as $\tilde{M}$ is a contact metric manifold too, we get:

$$d(\varphi^* \eta) = \varphi^* d\eta = \varphi^* \Phi = \pm (\cos \theta)\overline{\Phi}. \quad \text{... (2.7)}$$

But, if $\theta \neq 0$, then a contradiction follows from (2.6) and (2.7), and so, the result is proved. □

The following corollary gives an answer to our problem for three-dimensional slant submanifolds.

Corollary 2.7 — Let M be a three-dimensional slant submanifold of a Sasakian manifold $\tilde{M}$. Then, the Sasakian structure of $\tilde{M}$ induces a contact metric structure on M if and only if M is an invariant manifold.

PROOF : The direct implication follows directly from Theorem 2.6. The converse is well-known. □

Nevertheless, we can consider slant immersions between almost contact metric manifolds. In fact, it is enough to choose a local orthonormal frame $\{e_1, e_2, \xi\}$ and define $\overline{\phi}$ such that $\overline{\phi}e_1 = e_2$ and $\overline{\phi}e_2 = -e_1$ to obtain an almost contact structure on a three-dimensional slant submanifold.

Now, let

$$\varphi : (M, \overline{g}) \to (\tilde{M}, \phi, \xi, \eta, g)$$

be an isometric immersion from a Riemannian manifold in an almost contact metric manifold such that $\xi \in TM$.

Lemma 2.8 — In the above conditions, suppose that dim $M = m + 1$ and let $\{e_1, \ldots, e_m, \xi\}$ be a local orthonormal frame of TM. Then, the immersion φ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$\sum_{i=1}^{m} g(\phi e_j, e_i)\, g(\phi e_k, e_i) = \lambda \delta_{jk}, \quad \text{... (2.8)}$$

for any $j, k = 1, \ldots, m$. Moreover, in this case, $\lambda = \cos^2 \theta$, where θ denotes the slant angle of the immersion.

PROOF : Suppose that φ is a slant immersion with angle θ. Then, for any unit tangent field X in $\mathcal{D}$, we have:

$$\sum_{i=1}^{m} g^2(\phi X, e_i) = \cos^2 \theta. \quad \text{... (2.9)}$$

Hence, if we put $X = e_j$ in (2.9), it follows

$$\sum_{i=1}^{m} g^2(\phi e_j, e_i) = \cos^2 \theta, \quad \text{... (2.10)}$$

from where we obtain (2.8) in the case $j = k$ with $\lambda = \cos^2 \theta$. Now then, suppose that $j \neq k$. Then, $X = (1/\sqrt{2})\,(e_j + e_k)$ is a unit local field perpendicular to ξ, from which, by using (2.9), we have:

$$\cos^2\theta = \frac{1}{2}\sum_{i=1}^{m} g^2(\phi e_j, e_i) + \frac{1}{2}\sum_{i=1}^{m} g^2(\phi e_k, e_i) + \sum_{i=1}^{m} g(\phi e_j, e_i)\, g(\phi e_k, e_i)$$

By virtue of (2.10), this implies (2.8) in the case $j \neq k$.

Conversely, by using (2.8) it is easy to see that φ is slant with slant angle $\cos^{-1}\sqrt{\lambda}$. □

Remark 2.9 : The Kaehlerian version of Lemma 2.8 can be found in [5].

We can now consider an almost contact metric structure on M,

$$(M, \overline{\phi}, \xi, \overline{\eta}, \overline{g})$$

such that $\varphi_{*x}\,\overline{\xi}_x = \xi_{\varphi(x)}$ for any $x \in M$. Denote by $\overline{\Phi}$ and Φ the fundamental 2-forms of M and $\tilde{M}$, respectively. We also consider on M the 2-form $\varphi^*\,\Phi$. We can prove the following theorem.

Theorem 2.10 — *In the above conditions, suppose that there exists a constant* $k \in [-1, 1]$ *such that* $\varphi sp{*}\Phi = k\overline{\Phi}$. *Then,* φ *is slant with slant angle* $\cos^{-1}|k|$.

PROOF : Since M is an almost contact metric manifold, we can choose a $\overline{\phi}$-basis in M given by $\mathcal{B} = \{u_1, \ldots, u_m, \overline{\phi}u_1, \ldots, \overline{\phi}\,u_m, \xi\}$, dim $M = 2m + 1$.

We want to evaluate

$$\sum_{i=1}^{m} g(\phi X, u_i) g(\phi Y, u_i) + \sum_{i=1}^{m} g(\phi X, \overline{\phi}u_i)\, g(\phi Y, \overline{\phi}u_i), \quad \ldots (2.11)$$

for any $X, Y \in \mathcal{B} - \{\xi\}$, in order to apply Lemma 2.8.

Now then, since $\varphi^*\Phi = k\,\overline{\Phi}$, we have

$$g(\phi X, u_i) = kg(\overline{\phi}X, u_i) = -\,kg(X, \overline{\phi}u_i), \quad \ldots (2.12)$$

meanwhile

$$g(\phi X, \overline{\phi}u_i) = kg(\overline{\phi}X, \overline{\phi}u_i) = kg(X, u_i), \quad \ldots (2.13)$$

since $\overline{\eta}(u_i) = 0$.

Hence, if we put (2.12) and (2.13) in (2.11), we get:

$$k^2\left(\sum_{i=1}^{m} g(X, \overline{\phi}u_i)\, g(Y, \overline{\phi}u_i) + \sum_{i=1}^{m} g(X, u_i)\, g(Y, u_i)\right). \quad \ldots (2.14)$$

By checking (2.14) according to the choise of X and Y in $\mathcal{B}$, it is easy to see that Lemma 2.8 holds with $\lambda = k^2$. Hence, φ is a slant immersion with slant angle $\cos^{-1}|k|$. □

Corollary 2.11 — Suppose that dim M = 3. Then φ is slant if and only if there exists a constant $k \in [-1, 1]$ such that $\varphi^*\Phi = k\,\overline{\Phi}$. Moreover, in this case, $|k| = \cos\theta$, where θ denotes the slant angle of the immersion.

PROOF : It follows directly from Proposition 2.5 and Theorem 2.10. □

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