Indian J. pure appl. Math., **31**(11) : 1421-1427, November 2000

n-COLOUR COMPOSITIONS

A. K. AGARWAL

Centre for Advanced Study in Mathematics, Panjab University, Chandigarh 160 014, *India*

(*Received* 4 *November* 1999; *accepted* 22 *February* 2000)

Analogous to classical compositions we introduce here n-colour compositions and discuss combinatorial problems of which these new compositions are a solution. Other results obtained here include various generating functions, explicit formulas, recurrence relations and two computer produced tables. In Section 3 an n-colour composition function of a very general nature is defined. It is shown how this generalized n-colour composition function generalizes the classical composition function. Many of the results obtained in Section 3 for this generalized n-colour composition function reduce in a particular case to their corresponding results for classical compositions found in the literature.

Key Words : Partitions; Compositions; Generating Relations; Explicit Formulas, Recurrence Relations; Combinatorial Problems

1. INTRODUCTION

In the classical theory of partitions compositions were first defined by P.A. MacMahon as ordered partitions. For example, there are 5 partitions and 8 compositions of 4. The partitions are 4, 31, 22, 21^2, 1^4 and the compositions are 4, 31, 13, 22, 21^2, 121, 1^22, 1^4.

An n-colour partition (or a partition with "n copies of n") is defined as a partition in which a part of size n can come in n-different colours denoted by the subscripts $n_1, n_2, \ldots, n_n$ (cf. [1, 2]). Thus, for example, there are 13 n-colour partitions of 4 viz.,

4_1, 4_2, 4_3, 4_4

$3_1 1_1$, $3_2 1_1$, $3_3 1_1$

$2_1 2_1$, $2_1 2_2$, $2_2 2_2$

$2_1 1_1 1_1$, $2_2 1_1 1_1$, $1_1 1_1 1_1 1_1$

It was shown in [2] that if $P(\nu)$ denotes the number of n-colour partitions of ν then

$$1 + \sum_{\nu=1}^{\infty} P(\nu)\, q^{\nu} = \prod_{n=1}^{\infty} (1 - q^n)^{-n}. \qquad \ldots (1.1)$$

Since the right-hand side of (1.1) also generates plane partitions so the number of n-colour partitions of ν equals the number of its plane partitions. The n-colour partitions have connection with Baxter's solution of the hard-hexagon model [4, Chap. 14].

They were used in [1 & 2] to find several new Rogers-Ramanujan type identities. Analogous to P.A. MacMahon's definition of classical compositions we define an n-colour composition as follows :

Definition 1 — An n-colour ordered partition is called an n-colour composition.

Thus, for example, there are 21 n-colour compositions of 4, viz.,

$4_1, 4_2, 4_3, 4_4$

$3_1\, 1_1, 3_2\, 1_1, 3_3\, 1_1, 1_1\, 3_1, 1_1\, 3_2, 1_1\, 3_3$

$2_1\, 2_1, 2_1\, 2_2, 2_2\, 2_2, 2_2\, 2_1,$

$2_1\, 1_1\, 1_1, 2_2\, 1_1\, 1_1, 1_1\, 2_1\, 1_1, 1_1\, 1_1\, 2_1, 1_1\, 2_2\, 1_1, 1_1\, 1_1\, 2_2$

$1_1\, 1_1\, 1_1\, 1_1$

We shall denote the number of n-colour compositions of ν by $C(\nu)$ and the number of n-colour compositions of ν into m parts by $C(m, \nu)$. We see that $C(m, \nu)$ is the number of solutions of the following combinatorial problem :

Problem 1 — Let there be ν balls coloured with ν-colours such that the number of balls coloured with ith colour is not less than i. Suppose that each of these ν coloured balls has to be marked with m marks. How many different markings are possible, if there is atleast one ball with each mark?

We see that if we use the notation $[a]_b^j$ to denote that there are a balls with bth colour and jth mark, then the solution to the above problem is the number of solutions to the following equation:

$$[a_1]_{b_1}^1 + [a_2]_{b_2}^2 + \ldots + [a_m]_{b_m}^m = \nu, \qquad \ldots (1.2)$$

$$1 \le b_i \le a_i \le \nu,\ 1 \le i \le m,$$

which is $C(m, \nu)$.

Example — Consider the case $n = 4$ and $m = 2$. We get 10 solutions of (1.2), viz.,

$[3]_1^1 + [1]_1^2 = 4$

$[3]_2^1 + [1]_1^2 = 4$

$[3]_3^1 + [1]_1^2 = 4$

$[1]_1^1 + [3]_1^2 = 4$

$$[1]_1^1 + [3]_2^2 = 4$$

$$[1]_1^1 + [3]_3^2 = 4$$

$$[2]_1^1 + [2]_1^2 = 4$$

$$[2]_1^1 + [2]_2^2 = 4$$

$$[2]_2^1 + [2]_1^2 = 4$$

$$[2]_2^1 + [2]_2^2 = 4$$

Also, $C(2, 4) = 10$, since there are 10 *n*-colour compositions of 4 into 2 parts, viz.,

$$3_1\,1_1,\ ,\ 3_2\,1_1,\ ,\ 3_3\,1_1,\ ,\ 1_1\,3_1,\ ,\ 1_1\,3_2,\ ,\ 1_1\,3_3$$

$$2_1\,2_1,\ ,\ 2_2\,2_2,\ ,\ 2_1\,2_2,\ ,\ 2_2\,2_1.$$

In our next section we obtain generating relations and explicit formulas for $C(m, \nu)$ and $C(\nu)$. In Section 3, we define an *n*-colour composition function of very general nature and obtain generating function, recurrence relation and two computer produced tables.

The combinatorial problem of which this generalized *n*-colour composition function is a solution is also given. It is shown how this function generalizes classical compositions in a non-trivial way. The results obtained in Section 3 reduce to their corresponding results for the classical compositions in a particular case. We conclude in Section 4 by posing two open problems.

2. Generating Functions and Explicit Formulas

We give generating functions and explicit formulas for $C(m, \nu)$ and $C(\nu)$ in the form of the following result :

***Theorem* 1** — *Let $C(m, q)$ and $C(q)$ denote the enumerative generating functions for $C(m, \nu)$ and $C(\nu)$, respectively. Then*

$$C(m; q) = \frac{q^m}{(1-q)^{2m}}, \qquad \text{... (2.1)}$$

$$C(q) = \frac{q}{1 - 3q + q^2}, \qquad \text{... (2.2)}$$

$$C(m, \nu) = \binom{\nu + m - 1}{2m - 1}, \qquad \text{... (2.3)}$$

and

$$C(\nu) = F_{2\nu}, \qquad \text{... (2.4)}$$

where $F_{2\nu}$ is the (2ν)th Fibonacci number.

PROOF : Definition 1 implies that an n-colour composition is a composition in which each part of size n can appear in n-different colours denoted by the subscripts $n_1, n_2, \ldots, n_n$. In other words an n-colour composition is a weighted composition with weights 1, 2, 3, ...

Thus,

$$C(m; q) = \sum_{\nu=1}^{\infty} C(m, \nu)\, q^{\nu}$$

$$= (q + 2q^2 + 3q^3 + \ldots)^m$$

$$= \frac{q^m}{(1-q)^{2m}}.$$

This proves (2.1). On equating the coefficients of q^{ν} in (2.1), we get (2.3).

Now,

$$C(q) = \sum_{m=1}^{\infty} C(m; q)$$

$$= \sum_{m=1}^{\infty} \frac{q^m}{(1-q)^{2m}}$$

$$= \frac{q}{1 - 3q + q^2}.$$

which proves (2.2). Since the right-hand side of (2.2) also generates F_{2n}-the $(2n)$th Fibonacci numbers, we see that (2.4) is also proven. This completes the proof of Theorem 1.

3. A GENERALIZED n-COLOUR COMPOSITION FUNCTION

Let $C(r, k, m, \nu)$ denote the number of n-colour compositions of ν with exactly m parts, no part is greater than k and no subscript is greater than r.

We see that as $C(m, \nu)$ was the solution of the Problem 1, $C(r, k, m, \nu)$ is the solution of the following combinatorial problem :

Problem 2 — Let there be ν balls coloured with r colours $(r \le \nu)$ and the number of balls coloured with ith colour is not less than i and does not exceed k. Each of these balls is marked with m marks. How many different markings are possible, if there is atleast one ball of each mark?

Obviously, the solution of the above problem is the number of solutions to the following equation :

$$[a_1]_{b_1}^{1} + [a_2]_{b_2}^{2} + \ldots + [a_m]_{b_m}^{m} = \nu,\ 1 \le b_i \le \min(r, a_i),\ 1 \le a_i \le k,\ 1 \le i \le m \qquad \ldots (3.1)$$

Example — Consider the case $v = 4$, $m = 3$, $k = 2$ and $r = 2$. We get 6 solutions to (3.1), viz.,

$$[2]_1^1 + [1]_1^2 + [1]_1^3 = 4$$

$$[1]_1^1 + [2]_1^2 + [1]_1^3 = 4$$

$$[1]_1^1 + [1]_1^2 + [2]_1^3 = 4$$

$$[2]_2^1 + [1]_1^2 + [1]_1^3 = 4$$

$$[1]_1^1 + [2]_2^2 + [1]_1^3 = 4$$

$$[1]_1^1 + [1]_1^2 + [2]_2^3 = 4$$

Also, $C(2, 2, 3, 4) = 6$, since there are 6 n-colour compositions of 4 into 3 parts each part ≤ 2 and each subscript ≤ 2, viz.,

$$2_1\, 1_1\, 1_1,\ ,\ 1_1\, 2_1\, 1_1,\ ,\ 1_1\, 1_1\, 2_1$$

$$2_2\, 1_1\, 1_1,\ ,\ 1_1\, 2_2\, 1_1\ ,\ 1_1\, 1_1\, 2_2$$

If $C_1\,(r, k, m; q)$ is the enumerative generating function for $C(r, k, m, v)$, then

$$C(r, k, m; q) = \sum_{v \geq 1} C(r, k, m, v)\, q^v$$

$$=(q + 2q^2 + \ldots + rq^r + rq^{r+1} + \ldots + rq^k)^m \qquad \ldots (3.2)$$

Using (3.2) the following table (Table 1) of $C\,(r, k, m; q)$ for $m = 2$ and $1 \leq r, k \leq 4$ was obtained on MACSYMA - a Mathematics Software. We remark that for small values of m, r and k, as in the present case, such tables can be obtained without resorting to MACSYMA but simply by doing elementary arithmetic.

TABLE I

$r \backslash k$	1	2	3	4
1	q^2	$q^2 + 2q^3 + q^4$	$q^2 + 2q^3 + 3q^4 + 2q^5 + q^6$	$q^2 + 2q^3 + 3q^4 + 4q^5 + 3q^6 + 2q^7 + q^8$
2	q^2	$q^2 + 4q^3 + 4q^4$	$q^2 + 4q^3 + 8q^4 + 8q^5 + 4q^6$	$q^2 + 4q^3 + 8q^4 + 12q^5 + 12q^6 + 8q^7 + 4q^8$
3	q^2	$q^2 + 4q^3 + 4q^4$	$q^2 + 4q^3 + 10q^4 + 12q^5 + 9q^6$	$q^2 + 4q^3 + 10q^4 + 18q^5 + 21q^6 + 18q^7 + 9q^8$
4	q^2	$q^2 + 4q^3 + 4q^4$	$q^2 + 4q^3 + 10q^4 + 12q^5 + 9q^6$	$q^2 + 4q^3 + 10q^4 + 20q^5 + 25q^6 + 24q^7 + 16q^8$

If we write

$$C(r, k, \infty, v) = \sum_{m=1}^{\infty} C(r, k, m, v), \qquad \text{... (3.3)}$$

and

$$C(r, k, \infty; q) = \sum_{m=1}^{\infty} C(r, k, m; q), \qquad \text{... (3.4)}$$

then

$$C(r, k, \infty; q) = \sum_{m=1}^{\infty} C(r, k, m; q)$$

$$= \frac{q(1-q^r) - (1-q)\, rq^{k+1}}{(1-q)^2 - q\,(1-q^r) + (1-q)\, rq^{k+1}} \qquad \text{... (3.5)}$$

The following MACSYMA produced table (Table II) which was obtained by using (3.5) gives the first five terms of $C(r, k, \infty; q)$ for $1 \le r, k \le 5$:

TABLE II

r \ k	1	2	3	4	5
1	$q+q^2+q^3+q^4+q^5$	$q+2q^2+3q^3+5q^4+8q^5$	$q+2q^2+4q^3+7q^4+13q^5$	$q+2q^2+4q^3+8q^4+15q^5$	$q+2q^2+4q^3+8q^4+16q^5$
2	$q+q^2+q^3+q^4+q^5$	$q+3q^2+5q^3+11q^4+21q^5$	$q+3q^2+7q^3+15q^4+35q^5$	$q+3q^2+7q^3+17q^4+39q^5$	$q+3q^2+7q^3+17q^4+41q^5$
3	$q+q^2+q^3+q^4+q^5$	$q+3q^2+5q^3+11q^4+21q^5$	$q+3q^2+8q^3+17q^4+42q^5$	$q+3q^2+8q^3+20q^4+48q^5$	$q+3q^2+8q^3+20q^4+51q^5$
4	$q+q^2+q^3+q^4+q^5$	$q+3q^2+5q^3+11q^4+21q^5$	$q+3q^2+8q^3+17q^4+42q^5$	$q+3q^2+8q^3+21q^4+50q^5$	$q+3q^2+8q^3+21q^4+54q^5$
5	$q+q^2+q^3+q^4+q^5$	$q+3q^2+5q^3+11q^4+21q^5$	$q+3q^2+8q^3+17q^4+42q^5$	$q+3q^2+8q^3+21q^4+50q^5$	$q+3q^2+8q^3+21q^4+55q^5$

In the above table the coefficient of q^v is $C(r, k, \infty, v)$.

Remark : In view of the relationship

$$C(1, k, m, v) = C_m(k, v) \qquad \text{... (3.6)}$$

where $C_m(k, v)$ is the number of ordinary compositions of v with exactly m parts, no one of which is greater than k, we see that our generating function (3.5), (with $r = 1$) yields the following generating function for classical compositions with no part greater than k (cf. [3, eq. 940), p. 125]).

$$C(k; q) = \frac{q - q^{k+1}}{1 - 2q + q^{k+1}}.$$

We close this section with three recurrence relations in the form of a theorem.

Theorem 2 — *We have*

$$(1 - q)\, C(r, k, m; q) = \left(- rq^{k+1} + \sum_{j=1}^{r} q^j \right) C(r, k, m - 1; q), \qquad \dots (3.7)$$

$$C(r, k, m, v) - C(r, k, m, v - 1) = - r\, C(r, k, m - 1, v - k - 1)$$

$$+ \sum_{j=1}^{r} C(r, k, m - 1, v - j), \qquad \dots (3.8)$$

and

$$C(r, k, m, v) = \sum_{j=0}^{m} \binom{m}{j} r^j\, C(r, k - 1, m - j, v - kj). \qquad \dots (3.9)$$

PROOF : Easy and is hence omitted.

Remark : In view of the relationship (3.6) Theorem 2 reduces to its corresponding results for classical compositions found in the literature (cf. [3, Problem 11, p. 154]).

5. CONCLUSION

Many questions arise from this work. The most obvious among them are :

(1) Does Simon Newcomb's Problem admit a generalization in the light of *n*-colour compositions?

(2) What will be the shape of MacMahon's zig-zag graph in the case of *n*-colour compositions?

We conclude with the hope that like MacMahon's classical compositions these new combinatorial objects which we call *n*-colour compositions will find many more applications.

REFERENCES

1. A. K. Agarwal, *J. Number Theory* **28** (1988), 299-305.
2. A. K. Agarwal and G. E. Andrews, *J. Combin. Theory Ser. A* **45**, No. 1 (1987), 40-49.
3. J. Riordan, *An Introduction to Combinatorial Analysis*, Princeton University Press, 1978.
4. R. J. Baxter, *Exactly Solved Models in Statistical Mechaics*, Academic Press, London, 1982.