NUMERICAL SIMULATION OF THE EFFECT OF Polycythemia Vera ON HEAT FLOW IN HUMAN DERMAL REGIONS

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This paper deals with the numerical study of the effect of Polycythemia vera on the heat flow in human dermal regions. The Polycythemia vera is a type of blood cancer which is caused due to an abnormal increase in the concentration of red blood cells. With the increase in red blood cells the viscosity of the blood increases and thus the velocity of blood decreases. The changes in the blood flow affect the thermal conductivity and metabolic activity in the tissues and in turn it influences the heat flow in human body.

A model has been developed for one dimensional unsteady state case which leads to the initial boundary value problem. The human dermal region is nonhomogeneous and composite in nature. Therefore, this region is discretized into \( N \) layers to incorporate the variations in its significant parameters like blood mass flow rate, thermal conductivity and metabolic heat generation in the tissues. Also these parameters vary with respect to time due to Polycythemia vera. Appropriate boundary and initial conditions have been framed. The model is transformed into discretized variational form using a finite element approach. A computer program has been developed for the entire problem. The Euler's method has been employed for the time variable and Gauss elimination method has been used to solve the system of linear equations. As a special case, numerical simulation has been performed for \( (N = 3) \) three layered dermal region. The numerical results have been used to obtain temperature profiles in the region. The effect of Polycythemia vera on heat flow in human dermal region has been analyzed.

Key Words: Polycythemia vera; Blood Mass Flow Rate; Variational Form; Finite Element Method.

INTRODUCTION

Blood flow plays a significant role in temperature regulation of a human or animal body. Any abnormality in blood can influence temperature regulation process of the body. Polycythemia vera is a disease in which the viscosity of blood increases and thus, the increases and thus, the blood flow in vessels decreases. The blood flow serves as transport medium for nutrients, required by cells for metabolism. Thus any disturbance in blood flow will disturb the metabolic system. In Polycythemia vera, the decrease in blood flow will decrease the metabolic rate of the body. Further the thermal conductivity of the tissues is also affected by the changes in blood flow.

The mathematical model for heat flow in body tissues developed by Perl¹ is as given below:

\[
\rho \bar{c} \frac{\partial T}{\partial t} = \text{Div} (K \nabla T) + m_b c_b (T_A - T) + S, \tag{1}
\]

where \( \rho, \bar{c}, K \) and \( S \) are respectively the density, specific heat, thermal conductivity and rate of metabolic heat generation in tissues; \( m_b, c_b \) and \( T_A \) are the blood mass flow rate, specific heat of the blood and arterial blood temperature respectively. Perl¹ derived and used this equation to study
simple problems of heat flow in tissue medium, Patterson\textsuperscript{2,3} made an attempt for experimental
determination of temperature profiles in skin. Cooper and Trezek\textsuperscript{4,5}, Chao \textit{et al.}\textsuperscript{6,7}, Saxena\textsuperscript{8,9}
and Saxena \textit{et al.}\textsuperscript{10-13} had given a great deal of attention during the last few decades to study the
temperature distribution problems for a one dimensional and two dimensional steady and unsteady
state cases in a human body. They applied analytical and numerical methods to find solutions to
the above problems. In the above investigations, the effects of variation in blood mass flow rate,
metabolic heat generation in tissues, thermal conductivity, atmospheric temperature and rate of
evaporation on temperature distribution in skin and subcutaneous tissues have been investigated under
normal environmental and physiological conditions. Some attempts have also been made to study
these problems in skin and subcutaneous tissues involving abnormalities like tumors. Saxena and
Pardasani\textsuperscript{14}, Pardasani and Adlakha\textsuperscript{15} and Jain\textsuperscript{16} have also studied the temperature distribution in
SST region involving tumors for one dimensional steady state case.

In the above study the parameters like thermal conductivity, blood mass flow rate and
metabolic activity have been taken independent of time. However in some abnormal situations, like
\textit{Polycythemia vera} and blood cancer, these parameters will also depend on time variable. Here an
attempt has been made to study the effect of \textit{Polycythemia vera} on heat flow in human dermal
regions. The model has been developed for a one dimensional unsteady state case and finite element
approach has been used to obtain solution to the initial boundary value problem.

\textbf{MATHEMATICAL FORMULATION}

Eq. (1) for heat flow in SST region for a one dimensional unsteady state case can be rewritten as
given below :

\[
\frac{\partial}{\partial x} \left( K \frac{\partial T}{\partial x} \right) + M(T_\text{A} - T) + S = \rho c \frac{\partial T}{\partial t}, \quad \ldots (2)
\]

The surface is assumed to be exposed to the environment and hence boundary condition at
the outer surface can be written as:

\[
K \frac{\partial T}{\partial x} = h(T - T_a) + LE \text{ at } x = 0, \ t > 0, \quad \ldots (3)
\]

where \( h \) is heat transfer coefficient, \( T_a \) is atmospheric temperature, \( L \) and \( E \) are respectively the
latent heat & rate of sweat evaporation. The inner boundary condition is prescribed as given below

\[
T = T_b \text{ at } t \geq 0. \quad \ldots (4)
\]

The outer surface of skin is assumed to be insulated at time \( t = 0 \) and hence the initial
condition is given by

\[
T(x, 0) = T_b. \quad \ldots (5)
\]

The region is divided into \( N \) layers with different properties. The parameters like thermal
conductivity, blood mass flow rate and metabolic activity are taken as linearly variable with respect
to position. Apart from this these parameters are assumed to be decreasing function with respect to
time variable.
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\[ K^{(e)}(x) = \sum_{r=0}^{\tilde{m}} \alpha_r^{(e)} x^r, \quad \text{(6)} \]

\[ M^{(e)}(x) = \alpha e^{-\beta t} \sum_{r=0}^{\tilde{m}} \beta_r^{(e)} x^r, \quad \text{(7)} \]

and

\[ S^{(e)}(x) = \beta e^{-\xi} \sum_{r=0}^{\tilde{m}} \gamma_r^{(e)} x^r, \quad \text{(8)} \]

where \( \alpha_r^{(e)}, \beta_r^{(e)}, \) and \( \gamma_r^{(e)} \) are constants which determine physiological and physical properties of th element and \( \alpha, \beta, \gamma, \xi, \tilde{m}, \) \( \tilde{m} \) and \( \tilde{m} \) are constants. The following linear shape functions are assumed for \( T^{(e)} \) in each element \( x_i < x < x_j \)

\[ T^{(e)}(x, t) = c_1^{(e)} + c_2^{(e)} x \quad \text{(9)} \]

It can be written in the matrix form as

\[ T^{(e)} = p^T C^{(e)}, \quad \text{(10)} \]

where

\[ C^{(e)} = \begin{bmatrix} c_1^{(e)} \\ c_2^{(e)} \end{bmatrix}, \quad p^T = [1 \ x]. \]

Now

\[ T^{(e)} = T_i(t) \text{ at } x = x_i \]

\[ T^{(e)} = T_j(t) \text{ at } x = x_j \quad \text{(11)} \]

Using (9) and (11) we obtain

\[ \overline{T}^{(e)} = \overline{P}^{(e)} C^{(e)}, \quad \text{(12)} \]

where

\[ \overline{T} = \begin{bmatrix} T_i \\ T_j \end{bmatrix}, \quad \overline{P}^{(e)} = \begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix}. \]

From (12) we get

\[ C^{(e)} = R^{(e)} T(\overline{e}) \quad \text{(13)} \]

where

\[ R^{(e)} = P(\overline{e})^{-1} = \frac{1}{x_{ij}} \begin{bmatrix} x_j & -x_i \\ -1 & 1 \end{bmatrix} x_{ij} = x_j - x_i \]
From (10) and (13), we get

\[ T^{(e)} = \rho^T R^{(e)} T^{(e)} \]  

... (14)

Now we define

\[ \bar{T}^{(e)} = D^{(e)T} T, \]  

... (15)

where

\[ D^{(e)T} = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \end{bmatrix} 2 \times (n + 1) \]  

... (16)

and

\[ \bar{T} = [T_0, T_1, \ldots, T_i, T_j, \ldots, T_n]^T \] \( (n+1) \times 1 \) .

The discretized variational form of eqs. (2) and (3) can be written as:

\[ I^{(e)} = \frac{1}{2} \int_{x_i}^{x_f} \left[ K^{(e)} \left( \frac{\partial T^{(e)}}{\partial x} \right)^2 + M^{(e)} (T_A - T^{(e)})^2 - 2S^{(e)} T^{(e)} + \rho \bar{c} \frac{\partial (T^{(e)})^2}{\partial t} \right] dx \]

\[ + \left[ \frac{h}{2} (T^{(e)} - T_a)^2 + 2 \text{LET}^{(e)} \right] \]

... (17)

for \( e = 1, 2, \ldots, n \), where the term outside the integral holds for \( e = 1 \) only.

Now \( I^{(e)} \) can be written as:

\[ I^{(e)} = I_k^{(e)} + I_m^{(e)} + I_s^{(e)} + I_{\rho}^{(e)} + I_{\delta}^{(e)} , \]

... (18)

where

\[ I_k^{(e)} = \frac{1}{2} \int_{x_i}^{x_f} K^{(e)} \left( \frac{\partial T^{(e)}}{\partial x} \right)^2 , \]

... (19)

\[ I_m^{(e)} = \frac{1}{2} \int_{x_i}^{x_f} M^{(e)} (T_A - T^{(e)})^2 dx , \]

... (20)

\[ I_s^{(e)} = \int_{x_i}^{x_f} S^{(e)} T^{(e)} dx , \]

... (21)

\[ I_{\rho}^{(e)} = \frac{1}{2} \int_{x_i}^{x_f} \rho \bar{c} \frac{\partial (T^{(e)})^2}{\partial t} dx \]

... (22)

and

\[ I_{\delta}^{(e)} = 1/2 \left[ h (T^{(e)} - T_a)^2 + 2 \text{LET}^{(e)} \right]_{e=1} \]

... (23)
The arterial temperature is taken as:

\[ T_A^{(e)} = \sigma_1^{(e)} T_i + \sigma_2^{(e)} T_p, \]

where \( \sigma_2^{(e)} = x_1/x_3, \sigma_1^{(e)} = 1 - \sigma_2^{(e)}. \)

The integrals (17) are assembled to obtain:

\[ I = \sum_{e=1}^{N} I^{(e)}. \]

Now \( I \) is extremized with respect to each nodal temperature as given below:

\[ \frac{dI}{dT} = [0]_{(N+1) \times 1}, \]

where

\[ \frac{dI}{dT} = \left[ \frac{\partial I}{\partial T_r} \right]_{r=0}^{N} \]

or

\[ \frac{dI}{dT} = \sum_{e=1}^{N} \frac{dI^{(e)}}{dT} \]

From eq. (18), we get:

\[ \frac{dT^{(e)}}{dT^{(e)}} = \frac{dT_k^{(e)}}{dT^{(e)}} + \frac{dT_m^{(e)}}{dT^{(e)}} - \frac{dT_s^{(e)}}{dT^{(e)}} + \frac{dT_p^{(e)}}{dT^{(e)}} + \frac{dT_\delta^{(e)}}{dT^{(e)}}, \]

\[ \frac{dT_k^{(e)}}{dT^{(e)}} = B^{(e)} T^{(e)}, \]

\[ \frac{dT_m^{(e)}}{dT^{(e)}} = G^{(e)} + H^{(e)} T^{(e)}, \]

\[ \frac{dT_s^{(e)}}{dT^{(e)}} = F^{(e)} + Q^{(e)} T^{(e)}, \]

\[ \frac{dT_p^{(e)}}{dT^{(e)}} = M^{(e)} \frac{dT^{(e)}}{dt}, \]

and

\[ \frac{dT_\delta^{(e)}}{dT^{(e)}} = F^{(e)} + N^{(e)} T^{(e)}, \]
where $\mathbf{B}^{(e)}, \mathbf{H}^{(e)}, \mathbf{Q}^{(e)}, \mathbf{M}^{(e)}$ and $\mathbf{N}^{(e)}$ are $(2 \times 2)$ matrices and $\mathbf{G}^{(e)}, \mathbf{F}^{(e)}$ and $\mathbf{P}^{(e)}$ are $(2 \times 1)$ matrices.

But
\[
\frac{dI}{dT} = \sum_{e=1}^{n} \mathbf{D}^{(e)} \frac{dI^{(e)}}{dT^{(e)}}
\]  \hspace{1cm} \ldots (34)

Hence from eq. (28) to (34), we obtain the following system of first order ordinary differential equations:
\[
\mathbf{M} \frac{dT}{dt} + \mathbf{Y}T = \mathbf{Z}.
\]  \hspace{1cm} \ldots (35)

with the help of any suitable numerical or analytical method the system (35) can be solved to obtain $T_0, T_1, T_2, \ldots, T_N$.

**SPECIAL CASE ($N = 3$)**

As a special case, the SST region has been divided into three natural components, epidermis dermis and subcutaneous tissues. According to the properties of SST region, $M$ and $S$ are assumed to be negligible in epidermis, variable in dermis and uniform in subcutaneous tissues. The thermal conductivity is taken as constant, but different in each element. These assumptions are the same as used by Saxena & Bindra\textsuperscript{15}. The values of $K, M, S$ are mathematically expressed as given in eqs. (6), (7) and (8), where $m = 0$ in epidermis ($e = 1$) and subcutaneous tissues ($e = 3$) and $m = 1$ in dermis. ($e = 2$). The constants $\alpha, \beta$ and $\gamma$ involved in (6), (7) and (8) are as given below :

(i) **Epidermis** ($e = 1$)

\[
\alpha_0^{(1)} = K_1, \beta_0^{(1)} = 0, \gamma_0^{(1)} = 0
\]

and

\[
\alpha_1^{(1)} = 0, \beta_1^{(1)} = 0, \gamma_1^{(1)} = 0.
\]

(ii) **Dermis** ($e = 2$)

\[
\alpha_0^{(2)} = K_2, \beta_0^{(2)} = -mx_1/(x_2 - x_1)
\]

\[
\gamma_0^{(2)} = -Sx_1/(x_2 - x_1), \alpha_1^{(2)} = 0
\]

\[
\beta_1^{(2)} = m(x_2 - x_1), \gamma_1^{(2)} = S/(x_2 - x_1), q_e = 2/(T_a + T_b)
\]

(iii) **Subcutaneous Tissues** ($e = 3$)

\[
\alpha_0^{(3)} = K_3, \beta_0^{(3)} = m, \gamma_0^{(3)} = S, q_e = 1/T_b
\]

\[
\alpha_1^{(3)} = 0, \beta_1^{(3)} = 0, \gamma_1^{(3)} = 0
\]
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Using these assumptions, we get a system of ordinary differential equations (35). Further using initial condition, system (35) is solved using Euler's method and we get a system of linear equations

\[ \overline{C} \overline{T}_{j+1} = \overline{A} \] \[ j = 0, 1, 2 \ldots \] \[ \ldots (36) \]

where \( \overline{C}, \overline{T}_{j+1}, \overline{A} \) are matrices of order \( 3 \times 3 \), \( 3 \times 1 \) and \( 3 \times 1 \) respectively. The above system is solved using Gaussian elimination method. A computer program has been developed for the entire problem and executed to obtain numerical results.

**NUMERICAL RESULTS AND DISCUSSION**

The computations have been performed for two cases of atmospheric temperatures \( T_a = 15^\circ C \) and \( 23^\circ C \). The values of \( M, S \) and \( E \) have been taken accordingly from the table given below (Saxena & Bindra\(^2\)).

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Atmospheric Temp. ( ^\circ C )</th>
<th>Values of ( \text{Cal/cm}^2 \cdot \text{min.} )</th>
<th>Values of ( M ) ( \text{Cal/cm}^2 \cdot \text{min.} \cdot \text{deg.} \cdot \text{C} )</th>
<th>Values of ( \text{gms/cm}^2 \cdot \text{min.} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>0.0357</td>
<td>0.003</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>0.018</td>
<td>0.018</td>
<td>0.0</td>
</tr>
</tbody>
</table>

\[ \text{Values of gms/cm}^2 \cdot \text{min.} = 0.24 \times 10^{-3} \]

The values of other parameters are taken as given below (Saxena & Bindra\(^2\));

\( K_1 = 0.030 \text{ Cal/cm} \cdot \text{min.} \cdot \text{deg.C.} \)
\( K_2 = 0.045 \text{ Cal/cm} \cdot \text{min.} \cdot \text{deg.C.} \)
\( K_3 = 0.060 \text{ Cal/cm} \cdot \text{min.} \cdot \text{deg.C} \)
\( h = 0.009 \text{ Cal/cm}^2 \cdot \text{min.} \cdot \text{deg.C} \)
\( \overline{c} = 0.830 \text{ Cal/gm} \cdot \text{deg.C.} \)
\( \overline{\rho} = 1.090 \text{ gms/cm}^2 \).
\( L = 579.0 \text{ Cal/gm.} \)
\( T_b = 37 \text{ ^\circ C.} \)

The constants \( x_i \) \((i = 0(1)3)\) can be assigned any values depending on the sample of SST region under study. For a particular case of SST structure, following values have been assigned to \( x_i \) :

\[ x_0 = 0.0 \text{ cm}, \quad x_1 = 0.1 \text{ cm}, \quad x_2 = 0.35 \text{ cm}, \quad x_3 = 0.50 \text{ cm}, \]

\( \overline{\xi}, \overline{\alpha}, \overline{\gamma} \) are nonnegative constants representing ratio of respective processes in tissues of different types. In this case of Polycythemia vera, these constants are taken as
\[ \bar{\xi}_1 = K_1, \bar{\xi}_2 = K_1/2, \bar{\alpha} = \bar{\gamma} = 1 \] and the values of \( \theta, \beta \) and \( \delta \) are taken as:

\[ \bar{\theta} = \bar{\beta} = \bar{\delta} = 0.1, \text{ and } \bar{\theta} = \bar{\beta} = \bar{\delta} = 0, \] representing a normal subject without any disease. The step length \( h = 0.05 \) has been used in Euler’s method to obtain the numerical values.

Graphs have been plotted between nodal temperatures \( T_i \) (i = 0(1)2) and time, for the two cases of atmospheric temperatures, \( T_a = 15 \, ^{\circ}\text{C} \) and \( 23 \, ^{\circ}\text{C} \) and for different values of \( E, \bar{\theta}, \bar{\beta} \) and \( \bar{\delta} \). Fig. (1) is for \( T_a = 15 \, ^{\circ}\text{C} \) and \( E = 0 \). Figs. (2) and (3) are for \( T_a = 23 \, ^{\circ}\text{C} \) and \( E = 0, 0.24 \times 10^{-3} \, \text{gm/cm}^2\text{-min} \) respectively.

On comparing Figs. (1) and (2), it can be seen that fall in temperature profiles is more at lower atmospheric temperatures and same value of \( E = 0 \). At time \( t = 0 \), the skin surface is assumed to be perfectly insulated and therefore, each nodal temperature is at \( 37 \, ^{\circ}\text{C} \). As the insulation is released, the temperature profiles start falling and reach the steady state after some time. In the beginning, the profiles fall down quickly for \( \bar{\theta} = \bar{\beta} = \bar{\delta} = 0 \) (normal case) and then attain steady state in some time. Here \( \theta = \beta = \delta = 0 \) implies normal case i.e., the blood mass flow rate and metabolic heat generation are constant with respect to time.

From the figures (1), (2) and (3) the following observations are made:-

(i) For normal case, the time required by nodal temperatures to achieve steady state is higher at low atmospheric temperature. Also this steady state time increases with increase in rate of evaporation. This may be due to the higher temperature gradients for low atmospheric temperatures and the increase in heat loss at the surface due to sweat evaporation at the surface.

(ii) In case of Polycythemia vera, the steady state time is increasing with increase in atmospheric temperatures and increase in rate of evaporation. This is due to the disturbances caused by decreasing thermal conductivity, blood flow and metabolic activity.

It can be seen in all the figures that fall in temperature profiles is more for the case of polycythemia vera. Thus it can be inferred from the above that Polycythemia vera can cause significant thermal disturbances in the SST region.

The results obtained for normal case are in agreement with the result obtained by Saxena, Arya and Bindra. The results obtained are also in agreement with the physiological facts. To obtain more useful results, the experimental values of some of the parameters \( \bar{\theta}, \bar{\beta} \) and \( \bar{\delta} \) etc. are needed. Also this model can be improved to study the problem for various conditions.

The finite element method has proved to be very powerful and versatile for such situations. Such mathematical models can be developed to extract the information regarding temperature variation in the human body which may prove to be useful to biomedical scientists for diagnosis and treatment. This field is a potential area of research for applications of mathematics and computer and offers many challenging opportunities for interaction among applied mathematicians and biomedical scientists for practical applications.

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FIG. 1. Graph between temperature $T_i$ ($i = 0(1)2$) and time $t$ for $T_0 = 15 \, ^\circ\text{C}$ and $E = 0 \, \text{gm/cm}^3$ - min. Thick continuous lines for $\theta = \beta = \delta = 0.1$, thick broken lines for $\theta = \beta = \delta = 0.1$, thin continuous lines for $\theta = \beta = \delta = 0.001$, and thin broken lines for $\theta = \beta = \delta = 0$. 
Fig. 2. Graph between temperature $T_i$ ($i = 0(1)2$) and time $t$ for $T_a = 23 \, ^\circ\text{C}$ and $E = 0 \, \text{gm/cm}^3 \cdot \text{min}$. Thick continuous lines for $\delta = \beta = \delta = 0.1$, thick broken lines for $\delta = \beta = \delta = 0.1$, thin continuous lines for $\delta = \beta = \delta = 0.001$, and thin broken lines for $\delta = \beta = \delta = 0$. 
FIG. 3. Graph between temperature $T_i$ ($i = 0(1)2$) and time $t$ for $T_a = 23$ °C and $E = 0$ gm/cm$^3$ · min. Thick continuous lines for $\overline{\theta} = \overline{\beta} = \overline{\delta} = 0.1$, thick broken lines for $\overline{\theta} = \overline{\beta} = \overline{\delta} = 0.1$, thin continuous lines for $\overline{\theta} = \overline{\beta} = \overline{\delta} = 0.001$, and thin broken lines for $\overline{\theta} = \overline{\beta} = \overline{\delta} = 0$. 

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