Indian J. pure appl. Math., **31**(12) : 1621-1632, December 2000

A BELLMAN'S EQUATION FOR MINIMIZING THE MAXIMUM COST

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(*Received* 5 *October* 1999; *accepted* 8 *February* 2000)

In this paper we consider the problem that consists in finding the minimum of the maximum of a scalar functional on a trajectory. We present a Hamilton-Jacobi-Bellman (HJB) equation associated to the optimal cost. The corresponding solution is defined in the viscosity sense and we prove that the optimal cost is the unique solution.

Key Words : Minimax Optimization Problems; Minimax Optimal Control; Quasi-variational Inequalities; Bellman Equation; Viscosity Solution; Numerical Solution

1. INTRODUCTION AND DESCRIPTION OF THE PROBLEM

1.1 *Description of the Problem*

We consider in the interval $[0, T]$ a dynamic system which evolves according to the ordinary differential equation

$$\frac{dy}{ds}(s) = g(s, y(s), \alpha(s)), \; 0 \le t \le s \le T, \qquad \dots (1)$$

with initial condition

$$y(t) = x \in \mathcal{R}^r. \qquad \dots (2)$$

The set of controls is denoted generically by $\mathcal{A}\ (t, s)$,

$$\mathcal{A}\ (t, s) = \{\alpha : [t, s] \mapsto A \subset \mathcal{R}^m : \alpha(\cdot) \text{ measurable}\}. \qquad \dots (3)$$

The optimal control problem consists in minimizing the functional $J(t, x, \alpha(\cdot))$, where

$$J : [0, T] \times \mathcal{R}^r \times \mathcal{A}(t, T) \mapsto \mathcal{R}$$

$$(t, x, \alpha(\cdot)) \mapsto J(t, x, \alpha(\cdot)) = \text{ess sup } \{f(s, y(s), \alpha(s)) : s \in [t, T]\}. \qquad \dots (4)$$

This problem arises, for example, when we want to minimize the maximum deviation of the controlled trajectories with respect to a given special trajectory. This differs from those problems usually considered in the optimal control literature, where an accumulated cost is minimized. As considering an accumulated cost is not always the best method to qualify a controlled system with a unique scalar parameter, problems of this type have received considerable interest in recent publications (see e.g. [1]-[11] and [14]-[20]).

The objective of this work is to establish a Hamilton-Jacobi-Bellman (HJB) equation associated to the value function V without using the concept of discontinuous Hamiltonian used in [9], where a HJB equation defined in terms of a discontinuous Hamiltonian is studied.

So, let us define the optimal cost :

$$V : [0, T] \times \mathcal{R}^r \mapsto \mathcal{R}$$

$$(t, x) \mapsto V(t, x) = \inf \{J(t, x, \alpha(\cdot)) : \alpha(\cdot) \in \mathcal{A}(t, T)\}. \qquad \dots (5)$$

The principal result of this paper is the following one: we present a HJB equation defined in terms of a continuous Hamiltonian and we prove that the optimal cost V is the unique solution in the viscosity sense of this HJB equation.

1.2 The Value Function V

1.2.1 Technical Assumptions

Let $BUC([0, T] \times \mathcal{R}^r \times \mathcal{A})$ be the set of the bounded and uniformly continuous functions on

$$[0, T] \times \mathcal{R}^r \times A.$$

We assume that f and g satisfy the following hypotheses :

- $g : [0, T] \times \mathcal{R}^r \times A \mapsto \mathcal{R}^r,\ g \in BUC([0, T] \times \mathcal{R}^r \times A),$

  $f : [0, T] \times \mathcal{R}^r \times A \mapsto \mathcal{R}^r, f \in BUC([0, T] \times \mathcal{R}^r \times A).$

- $\forall\ x, \hat{x} \in \mathcal{R}^r$

$$\left| \begin{array}{l} \| g(t, x, a) \| \le M_g, \\ \| g(t, x, a) - g(t, \hat{x}, a) \| \le L_g \| x - \hat{x} \|, \end{array} \right. \qquad \dots (6)$$

$$\left| \begin{array}{l} 0 \le f(t, x, a) \le M_f, \\ | f(t, x, a) - f(\hat{t}, \hat{x}, a) | \le L_f (| t - \hat{t} | + \| x - \hat{x} \|) \end{array} \right. \qquad \dots (7)$$

- A is compact.

1.2.2 Properties of the Value Function V

The following properties have been established by Barron in [3] and Baron-Ishii in [9]:

- $V \in Lip([0, T] \times \mathcal{R}^r),$

where $Lip(G)$ is the set of all functions bounded and Lipschitz continuous in G.

The value function V satisfies the following dynamic programming principle:

$$\forall\ t \in [0, T), x \in \mathcal{R}^r$$

$$V(t, x) = \inf_{\alpha \in \mathcal{A}(t, s)} \left\{ \max \left(\operatorname*{ess\,sup}_{\tau \in [t, s]} f(\tau, y(\tau), \alpha(\tau)), V(s, y(s)) \right) \right\}, \qquad \dots (8)$$

with final condition

$$V(T, x) = \min_{\alpha \in A} f(T, x, a). \qquad \dots (9)$$

2. THE HAMILTON-JACOBI-BELLMAN EQUATION

The aim of this section is to arrive to an equation of HJB type associated to the optimal cost of the original minimax problem.

We start from the dynamical programming principle associated to the minimax problem and proceeding formally we set the HJB equation. In a second step, after defining the solution of the HJB equation in the viscosity sense, we prove that the optimal cost V is a solution in this sense of the HJB equation. In section 2.3 we prove that the solution is unique.

Definition 2.1 — To simplify notation, we will define

$$J(t, s, x, \alpha(\cdot)) = \operatorname{ess\,sup} \{f(\tau, y(\tau), \alpha(\tau)) : \tau \in (t, s)\}. \qquad \dots (10)$$

We start from the basic dynamical programming equation (8)

$$V(t, x) = \min_{\alpha(\cdot) \in \mathcal{A}(t, t+\delta)} (\max (J(t, t + \delta, x, \alpha(\cdot)), V(t + \delta, y(t + \delta)))) \qquad \dots (11)$$

and "proceeding formally" we get the HJB equation

$$\min_{\alpha \in A} \left(\max \left(f(t, x, a) - V(t, x), \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) \cdot g(t, x, a) \right) \right) = 0. \qquad \dots (12)$$

To prove it in a strict form, we use viscosity techniques (see [12, 13] and [21, 22]).

2.1 Definition of HJB Equation in the Viscosity Sense

We consider the equation

$$\min_{a \in A} (\max (f(t, x, a) - u(t, x), L_a u)(t, x))) = 0, \qquad \dots (13)$$

where
$$(L_a u)(t, x) = \frac{\partial u}{\partial t}(t, x) + \frac{\partial u}{\partial x}(t, x) \cdot g(t, x, a). \qquad \dots (14)$$

Subsolutions and Supersolutions

Definition 2.2 — We will say that w is a subsolution in the viscosity sense of (13) if $\forall$ $\phi \in C^1((0, T) \times \mathcal{R}^r)$ such that $w - \phi$ has a maximum at (t, x), it is verified

$$\min_{a \in A} (\max (f(t, x, a) - w(t, x), (L_a \phi)(t, x))) \geq 0. \qquad \text{... (15)}$$

Definition 2.3 — We will say that s is a supersolution in the viscosity sense of (13) if $\forall\ \phi \in C^1((0, T) \times \mathcal{R}^r)$, such that $s - \phi$ has a minimum at (t, x), it is verified

$$\min_{a \in A} (\max (f(t, x, a) - s(t, x), (L_a \phi)(t, x))) \leq 0. \qquad \text{... (16)}$$

Definition 2.4 — We will say that $u \in C((0, T) \times \mathcal{R}^r$ is a viscosity solution of the equation (13) with final condition (9) if u is both a viscosity subsolution and a viscosity supersolution of (13) and in addition it verifies (9).

2.2 The Cost V as a Viscosity Solution

2.2.1 V as Viscosity Subsolution

Let $(t, x) \in (0, T) \times \mathcal{R}^r$ and $\phi \in C^1((0, T) \times \mathcal{R}^r)$ such that

$$\begin{cases} V(t, x) = \phi(t, x) \\ V(s, y) \leq \phi(s, y)(s, y) \in [0, T] \times \mathcal{R}^r. \end{cases} \qquad \text{... (17)}$$

We take $a \in A$ and we define a set of controls $\alpha_\delta(\cdot)$ such that

$$\alpha_\delta(s) = a \ \ \forall\ s \in [t, t + \delta]. \qquad \text{... (18)}$$

The following alternative holds

$$\left| \begin{array}{l} V(t, x) \leq f(t, x, a) \\ \text{or} \\ V(t, x) > f(t, x, a). \end{array} \right. \qquad \text{... (19)}$$

• If $V(t, x) \leq f(t, x, a)$, then we have

$$\max \left(f(t, x, a) - V(t, x), \frac{\partial \phi}{\partial t}(t, x) + \frac{\partial \phi}{\partial x}(t, x) \cdot g(t, x, a) \right) \geq 0. \qquad \text{... (20)}$$

If $V(t, x) > f(t, x, a)$, obviously, it is

$$\lim_{\delta \to 0} J(t, t + \delta, x, \alpha_\delta(\cdot)) = f(t, x, a),$$

and so, $\exists \delta_0(t, x, a) > 0$ such that $\forall \delta \leq \delta_0(t, x, a)$

$$J(t, t + \delta, x, \alpha_\delta(\cdot)) < V(t, x). \qquad \text{... (21)}$$

By the principle of dynamic programming (8), we have

$$V(t, x) \leq \max(J(t, t + \delta, x, \alpha_\delta(\cdot)), V(t + \delta, y(t + \delta))), \qquad \text{... (22)}$$

in consequence, (21) and (22) imply that

$$V(t, x) \le V(t+\delta, y(t+\delta)).$$

Then, from (17) we have

$$\phi(t+\delta, y(t+\delta)) - \phi(t, x) \ge V(t+\delta, y(t+\delta)) - V(t, x) \ge 0 \qquad \text{... (23)}$$

dividing by $\delta \to 0$ and taking limit $\delta \to 0$, we get

$$\frac{\partial \phi}{\partial t}(t, x) + \frac{\partial \phi}{\partial x}(t, x) \cdot g(t, x, a) \ge 0. \qquad \text{... (24)}$$

In consequence, we have

$$\max\left(f(t, x, a) - V(t, x), \frac{\partial \phi}{\partial t}(t, x) + \frac{\partial \phi}{\partial x}(t, x) \cdot g(t, x, a) \right) \ge 0. \qquad \text{... (25)}$$

From (20) and 25), as a is an arbitrary element of A, it holds that

$$\min_{a \in A}\left(\max\left(f(t, x, a) - V(t, x), \frac{\partial \phi}{\partial t}(t, x) + \frac{\partial \phi}{\partial x}(t, x) \cdot g(t, x, a) \right)\right) \ge 0. \qquad \text{... (26)}$$

2.2.2 V as Viscosity Supersolution

Let $\phi \in C^1((0, T) \times \mathcal{R}^r)$ such that at (t, x) it is realized a minimum of $V - \phi$, i.e.

$$\left| \begin{array}{l} V(t, x) = \phi(t, x), \\ V(s, y) \ge \phi(s, y)\ (s, y) \in (0, T) \times \mathcal{R}^r. \end{array} \right. \qquad \text{... (27)}$$

We will prove by *reductio ad absurdum* that

$$\min_{a \in A}\left(\max\left(f(t, x, a) - V(t, x), \frac{\partial \phi}{\partial t}(t, x) + \frac{\partial \phi}{\partial x}(t, x) \cdot g(t, x, a) \right)\right) \le 0. \qquad \text{... (28)}$$

So, let us suppose that

$$\min_{a \in A}\left(\max\left(f(t, x, a) - V(t, x), \frac{\partial \phi}{\partial t}(t, x) + \frac{\partial \phi}{\partial x}(t, x) \cdot g(t, x, a) \right)\right) \ge \eta > 0. \qquad \text{... (29)}$$

We define

$$Z_\eta = \left\{ a \in A : f(t, x, a) \le V(t, x) + \frac{\eta}{2} \right\}, \qquad \text{... (30)}$$

$$G(Z_\eta) = \{ g(t, x, a) : a \in Z_\eta \}. \qquad \text{... (31)}$$

Since $\forall\ a \in Z_\eta$ it holds that $\exists$ a probability measure μ (with support in Z_η) such that

$$\bar{g} = \int_{Z_\eta} g(t, x, a)\, d\mu(a), \quad \text{... (33)}$$

and in consequence, by virtue of (32),

$$\frac{\partial \phi}{\partial t}(t, x) + \frac{\partial \phi}{\partial x}(t, x) \cdot \bar{g} = \int_{Z_\eta} \left(\frac{\partial \phi}{\partial t}(t, x) + \frac{\partial \phi}{\partial x}(t, x) \cdot g(t, x, a) \right)$$

$$d\mu(a) \geq \int_{Z_\eta} \eta\, d\mu(a) = \eta. \quad \text{... (34)}$$

Let $\alpha^\nu(\cdot)$ be a sequence of minimizing controls such that

$$J(t, x, \alpha^\nu(\cdot)) \leq V(t, x) + \frac{1}{\nu^2}. \quad \text{... (35)}$$

W.l.g. we will suppose (eventually redefining each control policy $\alpha^\nu(\cdot)$ on sets of null measure) that

$$f(s, y(s), \alpha^\nu(s)) \leq J(t, x, \alpha^\nu(\cdot)) \;\; \forall \; s \in [t, T]. \quad \text{... (36)}$$

So, from (35), (36) we have $\forall \; s \in (t, T)$

$$f(s, y(s), \alpha^\nu(s)) \leq V(t, x) + \frac{1}{\nu^2}. \quad \text{... (37)}$$

By virtue of (6) and (7), we have

$$| f(t + \delta, y(t + \delta), a) - f(t, x, a) | \leq L_f (1 + M_g)\, \delta, \quad \text{... (38)}$$

then,

$$\forall \; s \in \left[t, t + \frac{1}{\nu} \right]$$

$$f(t, x, \alpha^\nu(s)) \leq V(t, x) + \frac{1}{\nu^2} + L_f (1 + M_g) \frac{1}{\nu}. \quad \text{... (39)}$$

So, by virtue of (39), $\forall \;\; \nu \geq \nu_0 = 1 + \left[\max \left(\frac{2}{\sqrt{\eta}}, \frac{4 L_f (1 = M_g))}{\eta} \right) \right]$ we have

$$f(t, x, \alpha^\nu(s)) \leq V(t, x) + \frac{\eta}{2} \quad \text{... (40)}$$

then,

$$\forall \nu \geq \nu_0, \; \forall \; s \in \left[t, t + \frac{1}{\nu} \right], \; \alpha^\nu(s) \in Z_n. \quad \text{... (41)}$$

By definition of J we have

$$J\left(t+\frac{1}{\nu}, y\left(t+\frac{1}{\nu}\right), \alpha^{\nu}(\cdot)\right) \leq J\left(t, x, \alpha^{\nu}(\cdot)\right);$$

then, by (35)

$$V\left(t+\frac{1}{\nu}, y\left(t+\frac{1}{\nu}\right)\right) \leq J\left(t+\frac{1}{\nu}, y\left(t+\frac{1}{\nu}\right), \alpha^{\nu}(\cdot)\right) \leq J\left(t, x, \alpha^{\nu}(\cdot)\right) \leq V(t, x)+\frac{1}{\nu^{2}} \quad \ldots (42)$$

From (27) we have

$$V\left(t, \frac{1}{\nu}, y\left(t+\frac{1}{\nu}\right)\right) \geq \phi\left(t+\frac{1}{\nu}, y\left(t+\frac{1}{\nu}\right)\right) \quad \ldots (43)$$

and

$$V(t, x)=\phi(t, x); \quad \ldots (44)$$

in consequence, from (42), (43)

$$\phi\left(t+\frac{1}{\nu}, y\left(t+\frac{1}{\nu}\right)\right) \leq \phi(t, x)+\frac{1}{\nu^{2}}. \quad \ldots (45)$$

Taking limit $\nu \rightarrow \infty$, we get

$$\lim_{\nu \rightarrow \infty} \frac{1}{\left(\frac{1}{\nu}\right)}\left(\phi\left(t+\frac{1}{\nu}, y\left(t+\frac{1}{\nu}\right)\right)-\phi(t, x)\right) \leq 0. \quad \ldots (46)$$

It is easy to check that there exists $\bar{g}$ defined by (eventually for a suitable chosen subsequence)

$$\bar{g}=\lim_{\nu \rightarrow \infty} \frac{1}{\nu} \int_{0}^{\frac{1}{\nu}} g\left(t, x, \alpha^{\nu}(s)\right) d s \quad \ldots (47)$$

and in consequence it holds that :

$$\lim_{\nu \rightarrow \infty} \frac{1}{\left(\frac{1}{\nu}\right)}\left(\phi\left(t+\frac{1}{\nu}, y\left(t+\frac{1}{\nu}\right)\right)-\phi(t, x)\right)=\frac{\partial \phi}{\partial t}(t, x)+\frac{\partial \phi}{\partial x}(t, x) \cdot \bar{g}. \quad \ldots (48)$$

Then, from (46) we get

$$\frac{\partial \phi}{\partial t}(t, x)+\frac{\partial \phi}{\partial x}(t, x) \cdot \bar{g} \leq 0. \quad \ldots (49)$$

From (41) it is easy to check that

$$\bar{g} \in \overline{C_0\, G(Z_\eta)} \quad \text{... (50)}$$

and then, from (32) we get

$$\frac{\partial \phi}{\partial t}(t, x) + \frac{\partial \phi}{\partial x}(t, x) \cdot \bar{g} \geq \eta. \quad \text{... (51)}$$

As (51) contradicts (49), we obtain (28).

2.3 *Uniqueness*

To obtain the uniqueness of solution of (13) we will prove the following comparison result.

Theorem 2.1 — *Let* s *be a supersolution and* w *a subsolution of (13) (both bounded in* $(0, T) \times \mathcal{R}^r$*), then* $s \geq w$.

PROOF : We will proceed by *reductio ad absurdum* to prove that $s \geq w$. So, let us suppose that

$$\bar{r} := \sup_{(t, x) \in (0, T) \times \mathcal{R}^r} (w, t(x) - s(t, x)) > 0. \quad \text{... (52)}$$

Let be δ such that

$$0 < \delta < \min\left(\frac{\bar{r}}{2}, \frac{\bar{r}}{32\, T M_g}\right), \quad \text{... (53)}$$

then, $\exists\, (\hat{t}, \hat{x}) \in (0, T) \times \mathcal{R}^r$ such that

$$r := w(\hat{t}, \hat{x}) - s(\hat{t}, \hat{x}) > \sup_{(t, x) \in (0, T) \times \mathcal{R}^r} (w, t(x) - s(t, x)) - \delta > \frac{\bar{r}}{2}. \quad \text{... (54)}$$

We take the hump function

$$\Psi(x, y, t, t') = -\frac{\| x - y \|^2}{\varepsilon^2} - \frac{| t - t' |^2}{\varepsilon^2} - \rho\, \frac{(T - t)}{t} - \frac{\bar{r}}{8}\left(1 - \frac{t}{T}\right) + 2\delta \cdot \xi(\| x - \hat{x} \|), \quad \text{... (55)}$$

where (denoting by M a bound of $|s|$ and $|w|$)

$$\rho < \min\left(M, \frac{\bar{r}\,\hat{t}}{8\,(T - \hat{t})}\right) \quad \text{... (56)}$$

and $\xi(\cdot)$ is a C^1 function with the following properties

$$\begin{cases} \xi(\tau) = 0 & \tau \in [0, 1], \\ \xi(\tau) = -1 & \tau \in [3, +\infty], \\ |\xi'(\tau)| \leq 1 & \tau \in [1, 3]. \end{cases} \quad \text{... (57)}$$

So,

$$\begin{cases} \nabla_x \Psi(x, y, t, t') = \dfrac{2}{\varepsilon^2}(y-x) + 2\delta \cdot \xi'(\| x - \hat{x} \|) \cdot \dfrac{1}{\| x - \hat{x} \|} \cdot (x - \hat{x}), \\ \nabla_y \Psi(x, y, t, t') = \dfrac{2}{\varepsilon^2}(x-y), \\ \dfrac{\partial \Psi(x, y, t, t')}{\partial t} = \dfrac{2}{\varepsilon^2}(t'-t) + \dfrac{\rho T}{t^2} + \dfrac{\bar{r}}{8T}, \\ \dfrac{\partial \Psi(x, y, t, t')}{\partial t'} = \dfrac{2}{\varepsilon^2}(t-t'). \end{cases} \quad \text{... (58)}$$

Let (x_0, y_0, t_0, t_0') be a point that realizes the minimum in $[0, T] \times \mathcal{R}^r$ of

$$s(t, x) - w(t', y) - \Psi(x, y, t, t'). \quad \text{... (59)}$$

As the functions s and w are bounded by M, we get

$$\frac{\| x_0 - y_0 \|^2}{\varepsilon^2} + \frac{| t_0 + t_0' |^2}{\varepsilon^2} + \frac{\rho(T - t_0)}{t_0} + \frac{\bar{r}}{8} + 2\delta \leq 4M, \quad \text{... (60)}$$

which implies $\begin{cases} t_0 \geq \dfrac{\rho T}{5M}, \\ \| x_0 - y_0 \| \leq 2\sqrt{M}\,\varepsilon, \\ | t_0 - t_0' | \leq 2\sqrt{M}\,\varepsilon. \end{cases}$ ... (61)

Also, from (54), (55) and (57) we can easily check that $\| x_0 - \hat{x} \| \leq 3$. Obviously, w.l.g. we can suppose that ε is small enough to have $\| x_0 - y_0 \| \leq 1$. In addition

$$s(t_0, x_0) - w(t_0', y_0) - \Psi(x_0, y_0, t_0, t_0') \leq s(t_0, x_0) - w(t_0, x_0) - \Psi(x_0, x_0, t_0, t_0). \quad \text{... (62)}$$

So, by virtue of (55), 61)

$$\frac{\| x_0 - y_0 \|^2}{\varepsilon^2} + \frac{| t_0 - t_0' |^2}{\varepsilon^2} \leq w(t_0', y_0) - w(t_0, x_0) \leq \omega_w(\varepsilon), \quad \text{... (63)}$$

where $\omega_w(\cdot)$ is the modulus of continuity of w in the set $[0, T] \times B(\hat{x}, 4)$.

The function (of (t', y))

$$w(t', y) + \Psi(x_0, y, t_0, t') \quad \text{... (64)}$$

has a maximum at (t_0', y_0), then from (15) we have

$$\min_{a \in A} \left(\max\left(f(t_0', y_0, a) - w(t_0', y_0), (L_a(-\Psi(x_0, \cdot, t_0, \cdot)))(t_0', y_0)\right)\right) \geq 0, \quad \text{... (65)}$$

or in other words, $\forall\ a \in A$

$$f(t_0', y_0, a) - w(t_0', y_0) \geq 0 \qquad \dots (66)$$

or

$$(L_a(-\Psi(x_0, \cdot, t_0, \cdot)))(t_0', y_0) \geq 0. \qquad \dots (67)$$

In a similar way, the function (of (t, x))

$$s(t, x) - \Psi(x, y_0, t, t_0') \qquad \dots (68)$$

has a minimum at (t_0, x_0), then from (16) we have

$$\min_{a \in A} (\max(f(t_0', y_0, a) - s(t_0, x_0), (L_a \Psi(\cdot, y_0, \cdot\, t_0'))(t_0, x_0))) \leq 0. \qquad \dots (69)$$

Let $\bar{a}$ be a control that realizes the minimum in (69), then

$$f(t_0, x_0, \bar{a}) - s(t_0, x_0) \leq 0 \qquad \dots (70)$$

and

$$(L_{\bar{a}} \Psi(\cdot, y_0, \cdot, t_0'))(t_0, x_0) \leq 0. \qquad \dots (71)$$

We must now analyze two cases:

$$\text{(i)}\ \ 0 \leq \max(f(t_0', y_0, \bar{a}) - w(t_0', y_0), (L_{\bar{a}}(-\Psi(x_0, \cdot, t_0, \cdot)))(t_0', y_0))$$

$$= f(t_0', y_0, \bar{a}) - w(t_0', y_0).$$

In this case, taking into account (70), we get

$$f(t_0, x_0, \bar{a}) - s(t_0, x_0) \leq 0 \leq f(t_0', y_0, \bar{a}) - w(t_0', y_0)$$

By virtue of (63) we have

$$\| x_0 - y_0 \| + | t_0 - t_0' | \leq 2\, \varepsilon \sqrt{\omega_w(\varepsilon)}$$

and this inequality implies that

$$w(t_0', y_0) - s(t_0, x_0) \leq f(t_0', y_0, \bar{a}) - f(t_0, x_0, \bar{a})$$

$$\leq L_f(\| x_0 - y_0 \| + | t_0 - t_0' |) \leq 2\, L_f\, \varepsilon \sqrt{\omega_w(\varepsilon)}.$$

Absurd, because by virtue of (56) we have $\rho \dfrac{(T - \hat{t})}{\hat{t}} \leq \dfrac{\bar{r}}{8}$ and in consequence

$$w(t_0', y_0) - s(t_0, x_0) \geq w(\hat{t}, \hat{x}) - s(\hat{t}, \hat{x}) + \Psi(\hat{x}, \hat{x}, \hat{t}, \hat{t}) - \Psi(x_0, y_0, t_0, t_0')$$

$$\geq w(\hat{t}, \hat{x}) - s(\hat{t}, \hat{x}) + \Psi(\hat{x}, \hat{x}, \hat{t}, \hat{t})$$

$$\geq \frac{\bar{r}}{2} - \rho \frac{(T - \hat{t})}{\hat{t}} - \frac{\bar{r}}{8}\left(1 - \frac{t}{T}\right)$$

$$\geq \frac{\bar{r}}{2} - \frac{\bar{r}}{8} - \frac{\bar{r}}{8} = \frac{\bar{r}}{4}.$$

We conclude that (66) cannot hold for $\bar{a}$. Then, from (67) we have:

$$\text{(ii)}\ \ 0 \leq \max\,(f(t_0', y_0, \bar{a}) - w(t_0', y_0), (L_{\bar{a}}(-\Psi(x_0, \cdot, t_0, \cdot)))(t_0', y_0))$$

$$= (L_{\bar{a}}(-\Psi(x_0, \cdot, t_0, \cdot)))(t_0', y_0).$$

In consequènce

$$(L_{\bar{a}}(-\Psi(x_0, \cdot, t_0, \cdot)))(t_0', y_0) \geq 0 \qquad \text{... (72)}$$

and

$$(L_{\bar{a}}\, \Psi(x_0, \cdot, t_0, \cdot))(t_0, x_0) \leq 0. \qquad \text{... (73)}$$

But from (14) and (58) we have

$$(L_{\bar{a}}\, \Psi(x_0, \cdot, t_0, \cdot))(t_0', y_0) = \frac{2}{\varepsilon^2}(y_0 - x_0, g(t_0', y_0, \bar{a})) + \frac{2}{\varepsilon^2}(t_0 - t_0') \qquad \text{... (74)}$$

$$(L_{\bar{a}}\, \Psi(\cdot, y_0, t_0'))(t_0, x_0) = \frac{2}{\varepsilon^2}(y_0 - x_0, g(t_0, x_0, \bar{a})) + \frac{2}{\varepsilon^2}(t_0 - t_0') + \frac{\rho T}{t_0^2} + \frac{\bar{r}}{8T}$$

$$+\ 2\delta \cdot \frac{\xi'(\| x_0 - \hat{x} \|)}{\| x_0 - \hat{x} \|} \cdot ((x_0 - \hat{x})\, g(t_0, x_0, \bar{a})). \qquad \text{... (75)}$$

As (63) implies that

$$\left| \frac{2}{\varepsilon^2}(y_0 - x_0, g(t_0', y_0, \bar{a})) - \frac{2}{\varepsilon^2}(y_0 - x_0, g(t_0, x_0, \bar{a})) \right| \leq 2\, L_g\, \omega_w(\varepsilon)$$

and from (53) we have that

$$\left| 2\delta \cdot \frac{\xi'(\| x_0 - \hat{x} \|)}{\| x_0 - \hat{x} \|} \cdot ((x_0 - \hat{x}), g(t_0, x_0, \bar{a})) \right| \leq 2\delta M_g \leq \frac{\bar{r}}{16\,T}$$

we have that the inequalities (72)-(75) imply that

$$\bar{r} \leq 32\, T L_g\, \omega_w(\varepsilon), \qquad \text{... (76)}$$

inequality which contradicts the supposition (52). Then, by *reductio ad absurdum*, $\forall$ $(t, x) \in [0, T] \times \mathcal{R}^r$ we have that

$$w(t, x) \leq s(t, x). \qquad \ldots (77)\square$$

Corollary 2.1 — Let v and w be two solutions of (13), then $v = w$.

PROOF : It follows trivially from Theorem 2.1. □

CONCLUSIONS

Here, we have established a Bellman's equation to characterize the optimal cost of the problem (analyzed by Barron - Ishii in [9]) of minimizing the maximum cost.

While the HJB equation presented in [9] makes use of the concept of discontinuous Hamilonians, the equation obtained in this work is established in terms of continuous Hamiltonians.

In addition, the equation obtained is consistent with the numerical procedure developed in [16], procedure that brings discrete solutions which converge to the optimal cost function of the continuous problem.

ACKNOWLEDGEMENTS

The authors would like to thank The University of Rosario and CONICET for the financial support given to the Project "Control and Optimization. Theory and Applications" (Grants 19/1028 and PMT-PICT 0142).

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