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EXISTENCE OF PERIODIC ORBITS OF THIRD KIND IN THE PHOTOGRAVITATIONAL RESTRICTED PROBLEM OF FOUR BODIES IN THREE-DIMENSIONAL COORDINATE SSTEM

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The existence of periodic orbits of third kind in the photogravitational restricted problem of four bodies in three-dimensional cooridnate system is established.

Key Words : Four-body Problem; Photogravitational Field; Periodic Orbits

1. INTRODUCTION

Bhatnagar[4] established the existence of periodic orbits of collision in the restricted problem of four bodies. Ahmad and Mandal[2] generalized this problem by considering perturbations in coriolis and centrifugal forces and Ahmad[1] generalized this problem by considering all the three finite masses to be the sources of radiation pressures and they[1, 2] established the existence of periodic orbits of first kind. Bhatnagar and Taqvi[6] established the existence of periodic orbits of collision in three-dimensional restricted problem of four bodies and Ahmad and Mandal[3] considered this problem in the presence of perturbations in coriolis and centrifugal forces and established the existence of periodic orbits of third kind.

This note have establishes the existence of periodic orbits of third kind in the three-dimensional restricted problem of four bodies considering all the three finite masses to be the source of radiation pressures.

2. EQUATIONS OF MOTION

Consider the motion of an infinitesimal mass P in the photo-gravitational field of the primaries P_1, P_2, P_3 of finite masses m_1, m_2, m_3 respectively in a plane on circular orbits and let the mass of P be so small that the triangular configuration is not changed.

Let C be the geometric centre of the triangular configuration $P_1 P_2 P_3$ and G be the centre of mass of the masses m_1, m_2, m_3 situated at P_1, P_2, P_3 respectively.

Let us take the line Gx parallel to CP_1 as x-axis and the line Gy perpendicular to Gx in the sense of rotation as y-axis and the line through G and perpendicular to the plane $P_1 P_2 P_3$ as z-axis. Let the coordinates of P be (x, y, z) and those of P_i as (x_i, y_i, z_i), $(i = 1, 2, 3)$ in the rotating system.

If q_1, q_2, q_3 be the radiation factors of the gravitational forces due to radiations, then the Hamiltonian H is given by

$$H=\frac{1}{2}\left(p_1^2+p_2^2+p_3^2\right)+(p_1 y-p_2 x)-q_1 \mu_1/r_1 - q_2 \mu_2/r_2 - q_3 \mu_3/r_3=C, \qquad \text{... (1)}$$

where p_1, p_2, p_3 are the corresponding generalized momenta and r_1, r_2, r_3 are the distances of p from P_1, P_2, P_3 respectively.

Let us shift the origin to C, the geometric centre of the equilateral triangle $P_1 P_2 P_3$, through parallel axes. Let (x_1, x_2, x_3) be the coordinates of P and $(\bar{x}_1, \bar{x}_2, 0)$ be the coordinates of the centre of mass referred to $Cx_1 x_2 x_3$ system. Then we have

$$(x_1, x_2, x_3)=(x+\bar{x}_1, y+\bar{x}_2, z).$$

In canonical form, the equations of motion are

$$\dot{x}_1 = H_{p_i}, \dot{p}_i=-H_{x_i}, \ (i = 1, 2, 3), \qquad \text{... (2)}$$

where the dot denotes differentiation with respect to t and the suffixes denote partial derivatives with respect to the corresponding variables; and

$$H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+p_1(x_2-\bar{x}_2)-p_2(x_1-\bar{x}_1)$$

$$- q_1 \mu_1/r_1-q_2 \mu_2/r_2-q_3 \mu_3/r_3=C \qquad \text{... (3)}$$

$$C=C(\mu_1, \mu_2, \mu_3) = C_0+C_1 \mu_1+C_2 \mu_2+C_3 \mu_3+O(\mu^2). \qquad \text{... (4)}$$

3. Regularization

For the elimination of the singularity at P_1, we apply the transformations used by Bhatnagar[5] which may be generated by

$$S=\left(A+\xi_1^2-\xi_2^2\right)p_1+2\,\xi_1\,\xi_2 p_2+\xi_3 p_3 \qquad \text{... (5)}$$

such that

$$x_i=S_{p_i}, \pi_i=S_{\xi_i}, \ (i = 1, 2, 3),$$

where π_i are the momenta associated with new coordinates ξ_i $(i = 1, 2, 3)$ and $A = 1/\sqrt{3}$. We also introduce a new independent variable τ instead of t given by

$$dt=r_1\,d\tau, \ \tau = 0 \text{ when } t = 0. \qquad \text{... (6)}$$

The equations of motion (2) become

$$\xi_i' = K_{\pi_i},\ \pi_i' = -K_{\xi_i}\ ,\ (i = 1,\ 2,\ 3), \qquad \dots (7)$$

where dash denotes differentiation with respect to τ and

$$K = r_1\,(H - C) = \frac{1}{8\,\xi^2} r_1\,\pi^2 + \frac{1}{2} r_1\,\xi_3^2 + \frac{1}{2} r_1\,(\pi_1\,\xi_2 - \pi_2\,\xi_1)$$

$$- A\,(\pi_1\,\xi_2 + \pi_2\,\xi_1) - (\pi_1\,\xi_1 - \pi_2\,\xi_2)\,\bar{x}_2 + (\pi_1\,\xi_2 + \pi_2\,\xi_1)\,\bar{x}_1 - q_1\,\mu_1 - q_2\,\mu_2\,r_1/r_2$$

$$- q_3\,\mu_3\,r_1/r_3 - r_1\,(c_0 + c_1\,\mu_1 + c_2\,\mu_2 + c_3\,\mu_3$$

$$+ \text{higher orders of small quantities}) \qquad \dots (8)$$

$$r_1^2 = \xi^4 + \xi_3^2\ ,$$

$$r_2^2 = 1 + \xi^4 + 3A\,(\xi_1^2 - \xi_2^2)\ - 2\xi_1\,\xi_2 + \xi_3^2\ ,$$

$$r_3^2 = 1 + \xi^4 + 3A\,(\xi_1^2 - \xi_2^2) + 2\xi_1\,\xi_2 + \xi_3^2\ ,$$

$$\xi^2 = \xi_1^2 + \xi_2^2\ ,$$

$$\pi^2 = \pi_1^2 + \pi_2^2\ ,$$

We suppose that $\mu_3 = \mu$, $\mu_2 = \alpha\,\mu$ so that $\mu_1 = 1 - \mu\,(1 + \alpha)$.

Where α is a significant constant.

We have

$$\bar{x}_1 = A\,(2 - 3\,\alpha)\,\mu - 3\,\mu/2,$$

and
$$\bar{x} = \mu\,(\alpha - 1).$$

Taking ξ_3 to be of the order of μ, we have

$$r_1 = \xi^2 + o\,(\mu^2).$$

The Hamiltonian (8) can be put into the form

$$K = K_0 + \mu\,K_1,$$

where
$$K_0 = \frac{1}{8\,\xi^2} r_1\,\pi^2 + \frac{1}{2}\,\pi_3^2\,r_1 + \frac{1}{2} r_1\,(\pi_2\,\xi_1 - \pi_1\,\xi_2 - 2C_0),$$

$$K_1 = -\frac{1}{4}(\alpha-1)(\pi_1\xi_1 - \pi_2\xi_2) - \frac{3}{4}A(\alpha+1)(\pi_2\xi_1 + \pi_1\xi_2)$$

$$+(1-I_1)(\alpha+1) - (1-I_2)\alpha r_1/r_2 - (1-I_3)r_1/r_3 + r_1(\alpha C_1' + C_2'), \quad \ldots (9)$$

$$C_0' = C_0 + C_1,\ C_1' = C_1 - C_2,\ C_2' = C_1 - C_3\ ,$$

$$q_i = 1 - I_i,\ I_i \ll 1.$$

The solutions corresponding to K_0 has been obtained by Bhatnagar[5] introducing the polar coordinates ξ, θ defined by

$$\xi_1 = \xi \cos\theta,$$

and

$$\xi_2 = \sin\theta. \quad \ldots (10)$$

With the help of solutions obtained from K_0, K_1 can be expressed in terms of canonical elements $l, g, h; L, G, H$. The canonical equations for the complete Hamiltonian $K = K_0 + \mu K_1$ form the basis for the general perturbation theory for the prolem considered in this note.

4. Existence of Periodic Orbits of Third Kind

The orbits will be periodic for $\mu \neq 0$ if the following Duboshin's conditions are satisfied :

$$\frac{\partial[R_1]}{\partial w_i} = 0,\ (i = 1, 2, 3), \quad \ldots (11)$$

and

$$\frac{\partial[R_1]}{\partial a_i} = 0,\ (i = 1, 2, 3), \quad \ldots (12)$$

$$D = D_1 \cdot D_2 \neq 0, \text{ (Ahmad and Mandal[3])} \quad \ldots (13)$$

We have

$$R_0 = a_1 d - 1 + I_1,$$

where

$$d^2 = a_3^2 - 2(a_2 + C_0).$$

$$\therefore\ D_2 = -1/d^2 \neq 0. \quad \ldots (14)$$

Taking zero order terms, we get

$$r_1^2 = a^2 + \xi_3^2\ ,$$

$$r_2^2 = 1 + a^2 + 3aA\cos 2\theta - a\sin 2\theta + \xi_3^2\ ,$$

$$2\theta = y_1 + y_2,$$

$$x_i = a_i,\ y_i = \eta_i^{(0)}\ \tau' + w_i,\ (i = 1, 2, 3)$$

$$R_1 = \frac{1}{2}(\alpha + 1)\ G \sin 2\ \theta - \frac{3}{2} A\ (\alpha + 1)\ G \cos 2\theta$$

$$-\ (1 - I_2)\ a\ \alpha/r_2 - (1 - I_3)\ a/r_3 + a\ (\alpha\ C_1' + C_2').$$

$$\Rightarrow\ \frac{\partial [R_1]}{\partial w_2} = \frac{1}{2}(\alpha - 1)\ G \cos 2\theta + \frac{3}{2} A\ (\alpha + 1)\ G \sin 2\theta$$

$$+\ (1 - I_2)\ (-\ 3aA \sin 2\theta - a \cos 2\theta)\ a\ \alpha/r_2^3$$

$$+\ (1 - I_3)\ (-\ 3aA \sin 2\ \theta + a \cos 2\theta)\ a/r_3^3. \qquad \text{... (15)}$$

For $2\theta = 0, \pi$ etc, we have

$$\frac{\partial [R_1]}{\partial w_2} = \frac{1}{2}(\alpha - 1)\ G\ (1 - I_2)\ a^2\ \alpha/2r + (1 - I_3)\ a^2/2r^3,$$

where $$r^2 = 1 + a^2 + 3aA.$$

Therefore,

$$\frac{\partial [R_1]}{\partial w_2} = 0 \ \text{ if } \ G = \frac{a^2}{r^3}\ [1 - (\alpha I_2 - I_3)/(\alpha - 1)].$$

For $2\theta = 0, \pi$ etc and $G = \frac{a^2}{r^3}[1 - (\alpha I_2 - I_3)/(\alpha - 1)]$,

we have

$$\frac{\partial^2 [R_1]}{\partial w_2^2} = \frac{3a^2}{2r^5}[(\alpha I_2 - I_3) - (\alpha + 1)] + \frac{3a^2}{2r^3}(\alpha I_2 + I_3) \neq 0. \qquad \text{... (16)}$$

Ahmad and Mandal[3] has shown that

$$R_1 = \lambda' + \mu'\ \xi_3^2\ ,$$

where λ', μ' are independent of ξ_3.

Hence, proceeding as in Ahmad and Mandal[3], we can show that

$$D_1 \neq 0. \qquad \text{... (17)}$$

Using (14) and (17) we find that the condition (13) is satisfied.

It can be easily seen that the other conditions for the existence of periodic orbits are also satisfied.

Therefore, there exists periodic orbits of third kind in the photogravitational restricted problem of four bodies.

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