INVERSION OF A CONVOLUTION TRANSFORM INVOLVING GENERALIZED BATEMAN'S FUNCTION

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The inversion integrals for the integral transformations involving the generalized Bateman's function $K_n^l(x)$ in the kernel have been obtained. The inversion integral is expressed in terms of Whittaker's function.

1 INTRODUCTION

In the case of some functions it is possible to invert a certain convolution transform by a similar convolution transform. As a result the inversion integrals for the integral transformations involving Legendre, Chebysher, Gegenbauer, simple Laguerre and generalized Laguerre polynomials as the kernel are known (Buschman 1962a, b; Erdelyi 1963; Khandekar 1965; Ta Li 1960; Widder 1963). Shrivastava (1966) has determined the same with Whittaker's function $M_{k,\mu}^{(x)}$ as the kernel. Bharatiya (1967) was successful in inverting the transform with Mittag Leffler's function whereas an inversion integral for the integral equation with Whittaker's function $W_{k,\mu}^{(x)}(x)$ has been obtained by Joshi (1974).

In this paper we have attempted the inversion of the convolution transforms with generalized Bateman's function as the kernel.

Taking $(n+l)$ and $(n-l-1)$ as non-negative integers including zero such that $\pm l > -\frac{1}{2}$, we write

$$F_1(x) = K_{2n}^{2l}(\tfrac{1}{2}x) \quad \ldots \quad (1.1)$$

and

$$F_2(x) = x^{-l-\frac{1}{2}} M_{n-3/2,-l}^{(-x)} \quad \ldots \quad (1.2)$$

where $K_n^l(x)$ is the generalized Bateman's function defined by

$$K_n^l(x) = \int_0^{\pi/2} (2\cos\theta)^l \cos(x\tan\theta - n\theta)d\theta \quad \ldots \quad (1.3)$$

$$\text{for} \quad l > -1$$

and $M_{k,\mu}^{(x)}$ is the Whittaker's function.

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Theorem 1—If

(i) $\left(\frac{d}{dy}\right)^3[e^{\frac{1}{2}y}f(y)]$ is sectionally continuous in $0 \leqslant x < x_1 < \infty$ and

(ii) $f''(0) = f'(0) = f(0) = 0$,

then

$$\int_0^x F_1(x-t)g(t)dt = f(x) \qquad \ldots \quad (1.4)$$

has the solution

$$g(t) = \frac{1}{A}\int_0^t F_2(t-y)e^{-\frac{1}{2}y}\left(\frac{d}{dy}\right)^3[e^{\frac{1}{2}y}f(y)]dy, \quad \text{for } 0 < x < x_1 \qquad \ldots \quad (1.5)$$

where
$$A = \sqrt{1-2}l(-1)^{n-2l-\frac{1}{2}}$$

2. Results Required in the Proof

We shall denote the Laplace transform

$$\int_0^\infty e^{-pt}f(t)dt = F(p), \quad p > 0 \qquad \ldots \quad (2.1)$$

symbolically by
$$f(t) \doteqdot F(p).$$

We have due to Chakraverti (1953)

$$e^{-\frac{1}{2}t}k_{2n}^{2l}(\tfrac{1}{2}t) = \frac{(-1)^{n-l-1}}{\lfloor n+l+1}\left(\frac{d}{dt}\right)^{n-l-1}(e^{-t}t^{n+l}) \qquad \ldots \quad (2{\cdot}2)$$

where $(n-l-1)$ and $(n+l)$ are non-negative integers including zero such that $2l > -1$

$$k_{2n}^{2l}(\tfrac{1}{2}t) \doteqdot \frac{(-1)^{n-l-1}(p-\frac{1}{2})^{n-l-1}}{(p+\frac{1}{2})^{n+l+1}} \qquad \ldots \quad (2.3)$$

From (2.2) it is also easily obtained that

$$k_{2n}^{2l}(-\tfrac{1}{2}t) \doteqdot \frac{(-1)^{n+l}(p+\frac{1}{2})^{n-l-1}}{(p-\frac{1}{2})^{n+l+1}} \qquad \ldots \quad (2.4)$$

The following results are known from Erdelyi (1954, pp. 131, 133, 215, 129) and will be used in the sequel.

$$\int_0^t f_1(u)f_2(t-u)du = g_1(p).g_2(p) \qquad \ldots \quad (2.5)$$

Where $$f_1(t) \doteqdot g_1(p)$$
and $$f_2(t) \doteqdot g_2(p)$$

$$t^n \doteqdot \frac{\lfloor n+1}{p^{n+1}}, \quad p > 0 \qquad \dots \quad (2.6)$$

$$t^{\mu-1} M_{k,\mu}^{(at)} \doteqdot \frac{a^{\mu+\frac{1}{2}} \lfloor 2\mu+1 \; (p-a/2)^{k-\mu-\frac{1}{2}}}{(p+a/2)^{k+\mu+\frac{1}{2}}}, \; \mu > -\tfrac{1}{2}. \qquad \dots \quad (2.7)$$

If $$f(t) \doteqdot g(p)$$
then $$e^{-\beta t} f(t) \doteqdot g(p+\beta). \qquad \dots \quad (2.8)$$

By a slight change in variable and then making use of (2.3), (2.5) and (2.7) we can easily get the following result.

$$\int_y^x F_1(x-t).F_2(t-y)dt = \frac{A}{2}(x-y)^2 e^{-\frac{1}{2}(x-y)}. \qquad \dots \quad (2.9)$$

3. Proof of the theorem

To prove our result we substitute the proposed value of $g(t)$ from (1.5) into the L.H.S. of (1.4), changing the order of integration and using the result (2.9) we obtain

$$\tfrac{1}{2} e^{-\frac{1}{2}x} \int_0^x (x-y)^2 \left(\frac{d}{dy}\right)^3 [e^{\frac{1}{2}y} f(y)] dy$$

Successive integration under the condition (ii) of the theorem establishes Theorem I.

Theorem II—If

(i) $(n-l-1)$ and $(n+l)$ are non-negative integers including zero such that $\pm l > -\frac{1}{2}$

(ii) $\left(\frac{d}{dy}\right)^3 [e^{\frac{1}{2}y} f(y)]$ is sectionally continuous for $0 < x \leqslant x_1 < \infty$ and

(iii) $f''(0) = f'(0) = f(0) = 0$

then $$\int_0^x K_{2n}^{2l}[\tfrac{1}{2}(t-x)] g(t) dt = f(x) \qquad \dots \quad (4.1)$$

has the solution

$$g(x) = \frac{1}{B} \int_0^x (x-t)^{-l-\frac{1}{2}} M_{n+3/2,-l}^{(x-t)} e^{-\frac{1}{2}y} \left(\frac{d}{dy}\right)^3 [e^{\frac{1}{2}y} f(y)] dy,$$
$$\text{for} \quad 0 < x < x_1 \qquad \dots \quad (4.2)$$

where $$B = (-1)^{n+l} \lfloor -2l+1.$$

The proof of this theorem is similar to that of Theorem I.

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