ABSOLUTE NÖRLUND SUMMABILITY FACTORS OF FOURIER SERIES

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(Communicated by R. S. Mishra, F.N.A.)

(*Received 27 December 1972; after revision 25 April 1973*)

Pati (1961) established a theorem for ordinary Nörlund summability of Fourier series. In the present paper, a similar result for absolute Nörlund summability of Fourier series has been established.

1. Let Σa_n be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex

$$P_n = p_0+p_1+p_2+\dots+p_n$$
$$P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} S_\nu$$

$$= \frac{1}{P_n} \sum_{\nu=0}^{n} P_\nu a_{n-\nu} \quad (P_n \neq 0) \qquad \dots \quad (1.1)$$

defines the sequence $\{t_n\}$ of Nörlund (1919) means of the sequence $\{S_n\}$ generated by the sequence of coefficients $\{p_n\}$.

The series Σa_n is said to be summable (N, p_n) to the sum S, if $\lim_{n\to\infty} t_n$ exists and is equal to S, and is said to be absolutely summable (N, p_n) (Mears 1935) or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation, that is

$$\Sigma\, |t_n - t_{n-1}| < \infty$$

or symbolically $\{t_n\} \in BV$.

2. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality the constant term in the Fourier series can be taken to be zero, so that

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$\equiv \sum_{n=1}^{\infty} A_n(t)$$

and
$$\int_{-\pi}^{\pi} f(t)dt = 0$$

we write
$$\phi(t) = \tfrac{1}{2}[f(x+t)+f(x-t)-2f(x)]$$
$$\phi^*(t) = \tfrac{1}{2}[f(x+t)+f(x-t)]$$
$$P_\tau = P_{[1/t]} \quad \text{and} \quad p_\tau = p_{[1/t]}$$

where $\tau = [1/t]$ denotes the integral part of $1/t$.

In 1961, Pati proved a theorem for ordinary Nörlund summability in the following form.

Theorem—If (N, p_n) be a regular Nörlund method, defined by a real, non-negative, monotonic non-increasing sequence of coefficients $\{p_n\}$, such that $P_n \to \infty$ and $\log n = O(P_n)$, as $n \to \infty$, then if
$$\Phi(t) \equiv \int_0^t |\phi(u)|\, du = o\left[\frac{t}{P_\tau}\right]$$

as $t \to +0$, where $\tau = [1/t]$, the Fourier series of $f(t)$ at $t = x$ is summable (N, p_n) to $f(x)$.

The question now naturally arises as to whether it is possible to establish similar result for absolute Nörlund summability of Fourier series. The answer to this question seems to be in affirmative. In an attempt to answer the question we are proving the following theorem in which we have taken a more general condition than that of Pati.

Theorem—If
$$\Phi^*(t) \equiv \int_0^t |\phi^*(u)|\, du = O\left[\frac{\beta\left(\frac{1}{t}\right) p_\tau}{\alpha(P_\tau)}\right]$$

as $t \to +0$, where $\beta(t)$ and $\alpha(t)$ are functions of t such that $\beta(t)$, $\alpha(t)$ and $\frac{\beta(t)t}{\alpha(t)}$ increase monotonically with t and
$$\beta(n)P_n = O[\alpha(p)_n]$$

as $n \to \infty$, then the factored Fourier series
$$\Sigma \frac{A_n(t)\lambda_n P_n}{n \log (n+1)}$$

is summable $|N, p_n|$ at $t = x$, where $\{\lambda_n\}$ is a convex sequence, such that $\Sigma n^{-1}\lambda_n$ is convergent and $\{p_n\}$ is a real, non-negative, monotonic non-increasing sequence of coefficients, such that $P_n \to \infty$.

4. We need the following lemmas for proving our theorem :

Lemma 1 (Chow 1941)—If $\{\lambda_n\}$ is a convex sequence, such that $\Sigma n^{-1}.\lambda_n$ is convergent then λ_n is non-negative and decreasing, $n\Delta\lambda_n \to 0$, $\lambda_n \log n = 0(1)$ as $n \to \infty$.

Lemma 2—If

$$\Phi^*(t) = o\left[\frac{\beta\left(\frac{1}{t}\right)p_\tau}{\alpha(P_\tau)}\right] \quad \text{as } t \to +0, \text{ then}$$

$$S_n(x) = 0(\log n)$$

as $n \to \infty$, where $S_n(x)$ denotes the n-th partial sum of the Fourier series.

Proof : We write

$$S_n(x) = \frac{1}{2\pi}\int_0^\pi \phi^*(u)\,\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du.$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{n}} \phi^*(u)\,\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du$$

$$+ \frac{1}{2\pi}\int_{\frac{1}{n}}^{\delta} \phi^*(u)\,\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du$$

$$+ \frac{1}{2\pi}\int_{\delta}^{\pi} \phi^*(u)\,\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du.$$

$$= R_1 + R_2 + R_3, \text{ say.}$$

Then we have

$$|R_1| \leqslant \frac{1}{\pi}\,n\int_0^{\frac{1}{n}} |\phi^*(u)|\,du$$

$$= o\left[\frac{n\beta(n)\,p_n}{\alpha(P_n)}\right]$$

$$= 0(1) \text{ as } n \to \infty.$$

$$|R_2| \leqslant \frac{1}{\pi}\int_{\frac{1}{n}}^{\delta} \frac{|\phi^*(u)|}{u}\,du$$

$$= \frac{1}{\pi}\left|\left[\frac{\Phi^*(u)}{u}\right]_{\frac{1}{n}}^{\delta} - \int_{\frac{1}{n}}^{\delta} \frac{\Phi^*(u)}{u^2}\,du\right|$$

$$\leqslant \frac{1}{\pi}\left|\left[\frac{\Phi^*(u)}{u}\right]_{\frac{1}{n}}^{\delta}+\frac{1}{\pi}\right|\int_{\frac{1}{n}}^{\delta}\frac{\Phi^*(u)}{u^2}\,du\Big|$$

$$=0(1)+O\left[\sum_{k=1}^{n-1}\int_{k}^{k+1}\Phi^*\left(\frac{1}{w}\right)dw\right]$$

but

$$\int_{k}^{k+1}\Phi^*\left(\frac{1}{w}\right)dw\leqslant\Phi^*\left(\frac{1}{k}\right)$$

$$=O\left[\frac{\beta(k)p_k}{\alpha(P_k)}\right]$$

so

$$|R_2|=o\left[\Sigma\frac{\beta(k)p_k}{\alpha(P_k)}\right]$$

$$=o\left[\sum_{k=1}^{n-1}\frac{\beta(k)p_k\,.\,k}{k\alpha(P_k)}\right]$$

$$=o\left[\frac{\beta(n)P_n}{\alpha(P_n)}\sum_{k=1}^{n}\frac{1}{k}\right]$$

$$=o\left[\frac{\beta(n)P_n\log n}{\alpha(P_n)}\right]=o\,(.\log n)$$

as $n\to\infty$, by the hypothesis of the theorem. And, by Riemann-Lebesgue theorem

$$|R_3|=o(1).$$

Lemma 3—Let $S_n(x)$ denote the n-th partial sum of the Fourier series. If

$$S_n(x)=0(\log n)$$

then

$$T_n\equiv\sum_{\nu=1}^{n}\nu A_\nu(x)=o(n\log n)$$

This lemma follows from Abel's transformation.

5. Proof of the Theorem—By using (1.1) we have

$$t_n-t_{n-1}=\sum_{\nu=0}^{n-1}\left(\frac{P_\nu}{P_n}-\frac{P_{\nu-1}}{P_{n-1}}\right)u_{n-\nu}.$$

$$=\frac{1}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}(P_n\,p_\nu-P_\nu\,p_n)u_{n-\nu}.$$

$$=\frac{1}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}[P_n(p_{n-\nu-1}-p_n)]u_{\nu+1}$$

$$+\frac{1}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}[p_n(P_n-P_{n-\nu-1})]u_{\nu+1}$$

$$=I+J,\ \text{say}$$

where

$$U_n=\frac{A_n(t)\lambda_nP_n}{n\log(n+1)}.$$

For the proof of the theorem we have to show that

$$\sum_{n=2}^{\infty}|t_n-t_{n-1}|\leqslant\sum_{n=2}^{\infty}|I|+\sum_{n=2}^{\infty}|J|<\infty. \qquad \dots\ (5.1)$$

Applying Abel's transformation to the summation part of I, we have

$$\sum_{n=2}^{\infty}|I|\leqslant\sum_{n=2}^{\infty}\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-2}\left|T_{\nu+1}\,\Delta\left\{\frac{(p_{n-\nu-1}-p_n)\lambda_{\nu+1}P_{\nu+1}}{(\nu+1)^2\log(\nu+2)}\right\}\right|$$

$$+\sum_{n=2}^{\infty}\left|\frac{T_n(p_0-p_n)\lambda_nP_n}{n^2\log(n+1)}\right|$$

$$=\sum_{n=2}^{\infty}|I_1|+\sum_{n=2}^{\infty}|I_2|. \qquad \dots\ (5.2)$$

$$\sum_{n=2}^{m}|I_1|=\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-2}\left|T_{\nu+1}\Delta\left\{\frac{(p_{n-\nu-1}-p_n)\lambda_{\nu+1}P_{\nu+1}}{(\nu+1)^2\log(\nu+2)}\right\}\right|$$

$$=o\left[\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-2}(p_{n-\nu-2}-p_{n-\nu-1})\frac{\lambda_{\nu+1}P_{\nu+1}T_{\nu+1}}{(\nu+1)^2\log(\nu+2)}\right]$$

$$+o\left[\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-2}(p_{n-\nu-1}-p_n)\frac{\Delta\lambda_{\nu+1}P_{\nu+1}T_{\nu+1}}{(\nu+1)^2\log(\nu+2)}\right]$$

$$+o\left[\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-2}(p_{n-\nu-1}-p_n)\frac{\lambda_{\nu+1}T_{\nu+1}p_{\nu+1}}{(\nu+1)^2\log(\nu+2)}\right]$$

$$+o\left[\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-2}(p_{n-\nu-1}-p_n)\frac{\lambda_{\nu+1}T_{\nu+1}P_{\nu+1}}{(\nu+1)^3\log(\nu+2)}\right]$$

$$+o\left[\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-2}(p_{n-\nu-1}-p_n)\frac{\lambda_{\nu+1}T_{\nu+1}P_{\nu+1}}{(\nu+1)^2(\nu+2)\log^2(\nu+2)}\right]$$

$$=o\left[\sum_{\nu=0}^{m-2}\frac{\lambda_{\nu+1}P_{\nu+1}}{\nu+1}\sum_{n=\nu+2}^{m}\frac{p_{n-\nu-2}-p_{n-\nu-1}}{P_{n-1}}\right]$$

$$+o\left[\sum_{\nu=0}^{m-2}\frac{\Delta\lambda_{\nu+1}}{\nu+1}P_{\nu+1}\sum_{n=\nu+2}^{m}\frac{p_{n-\nu-1}-p_{n-\nu}}{P_{n-1}}\right]$$

$$+o\left[\sum_{\nu=0}^{m-2}\frac{\lambda_{\nu+1}p_{\nu+1}}{\nu+1}\sum_{n=\nu+2}^{m}\frac{p_{n-\nu-1}-p_{n-\nu}}{P_{n-1}}\right]$$

$$+o\left[\sum_{\nu=0}^{m-2}\frac{\lambda_{\nu+1}P_{\nu+1}}{(\nu+1)^2}\sum_{n=\nu+2}^{m}\frac{p_{n-\nu-1}-p_{n-\nu}}{P_{n-1}}\right]$$

$$+o\left[\sum_{\nu=0}^{m-2}\frac{\lambda_{\nu+1}P_{\nu+1}}{(\nu+1)(\nu+2)\log(\nu+2)}\sum_{n=\nu+2}^{m}\frac{p_{n-\nu-1}-p_{n-\nu}}{P_{n+1}}\right]$$

$$=o\left[\sum_{\nu=0}^{m-2}\frac{\lambda_{\nu+1}}{\nu+1}\right]+0\left[\sum_{\nu=0}^{m-2}\frac{\Delta\lambda_{\nu+1}}{\nu+1}\right]+0\left[\sum_{\nu=0}^{m=2}\frac{\lambda_{\nu+1}}{(\nu+1)^2}\right] \qquad \ldots (5.3)$$

$$=o(1)$$

as $m\to\infty$, since

$$(p_{n-\nu-1}-p_n)\leqslant(\nu+1)(p_{n-\nu-1}-p_{n-\nu})$$

$$(\nu+1)p_{\nu+1}\leqslant P_{\nu+1}$$

and
$$\sum_{n=\nu+2}^{m}\frac{p_{n-\nu-1}-p_{n-\nu}}{P_{n-1}}=o\left(\frac{1}{P_{\nu+1}}\right)$$

as $m\to\infty$.

And

$$\sum_{n=2}^{\infty}|I_2|=\sum_{n=2}^{\infty}\left|T_n\frac{(p_0-p_n)\lambda_nP_n}{n^2\log(n+1)}\right|$$

$$=o\left[\sum_{n=2}^{\infty}\left|\frac{(p_0-p_n)\lambda_nP_n\,n\log n}{n^2\log(n+1)}\right|\right]$$

$$=o\left[\sum_{n=2}^{\infty}|(p_0-p_n)\frac{\lambda_nP_n\log n}{n\log(n+1)}\right]$$

$$=o(1). \qquad \ldots (5.4)$$

By virtue of (5.2)-(5.4) we get

$$\sum_{n=2}^{\infty}|I|=0(1). \qquad \ldots (5.5)$$

Similarly by Abel's transofrmation we have

$$\sum_{n=2}^{\sigma} |J| \leqslant \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-2} \left| T_{\nu+1} \Delta \left\{ \frac{(P_n - P_{n-\nu-1})\lambda_{\nu+1} P_{\nu+1}}{(\nu+1)^2 \log(\nu+2)} \right\} \right|$$

$$+ \sum_{n=2}^{\infty} \left| T_n \frac{p_n(P_n - P_0)\lambda_n P_n}{n^2 P_n P_{n-1} \log(n+1)} \right|$$

$$= \sum_{n=2}^{\sigma} |J_1| + \sum_{n=2}^{\sigma} |J_2|, \text{ say.} \qquad \dots (5.6)$$

Now

$$\sum_{n=2}^{m} |J_1| = \sum_{n=2}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-2} \left| \Delta \left\{ \frac{(P_n - P_{n-\nu-1})\lambda_{\nu+1} P_{\nu+1}}{(\nu+1)^2 \log(\nu+2)} \right\} T_{\nu+1} \right|$$

$$= o\left[\sum_{n=2}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{\nu-0}^{n-2} (p_{n-\nu-1} - p_{n-\nu-2}) \frac{\lambda_{\nu+1} P_{\nu+1} \log(\nu+1)}{(\nu+1)\log(\nu+2)} \right]$$

$$= o\left[\sum_{n=2}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-2} (P_n - P_{n-\nu-1}) \frac{\Delta\lambda_{\nu+1} P_{\nu+1} \log(\nu+1)}{(\nu+1)\log(\nu+2)} \right]$$

$$+ o\left[\sum_{n=2}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-2} (P_n - P_{n-\nu-1}) \frac{\lambda_{\nu+1} p_{\nu+1} \log(\nu+1)}{(\nu+1)\log(\nu+2)} \right]$$

$$+ o\left[\sum_{n=2}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-2} (P_n - P_{1-\nu-1}) \frac{\lambda_{\nu+1} P_{\nu+1} \log(\nu+1)}{(\nu+1)^2 \log(\nu+2)} \right]$$

$$+ o\left[\sum_{n=2}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-2} (P_n - P_{n-\nu-1}) \frac{\lambda_{\nu+1} P_{\nu+1} \log(\nu+1)}{(\nu+1)(\nu+2)\log^2(\nu+2)} \right]$$

$$= o\left[\sum_{\nu=0}^{m-2} \frac{\lambda_{\nu+1} \log(\nu+1)}{(\nu+1)\log(\nu+2)} P_{\nu+1} \sum_{n=\nu+2}^{m} p_{n-\nu-1} \frac{p_n}{P_n P_{n-1}} \right]$$

$$+ o\left[\sum_{\nu=0}^{m-2} \frac{\Delta\lambda_{\nu+1} \log(\nu+1)}{(\nu+1)\log(\nu+2)} P_{\nu+1} \sum_{n=\nu+2}^{m} p_{n-\nu} \frac{p_n}{P_n P_{n-1}} \right]$$

$$+ o\left[\sum_{\nu=0}^{m-2} \frac{\lambda_{\nu+1} \log(\nu+1)}{(\nu+1)\log(\nu+2)} p_{\nu+1} \sum_{n=\nu+2}^{m} p_{n-\nu} \frac{p_n}{P_n P_{n-1}} \right]$$

$$+ o\left[\sum_{\nu=0}^{m-2} \frac{\lambda_{\nu+1} P_{\nu+1} \log(\nu+1)}{(\nu+1)^2 \log(\nu+2)} \sum_{n=\nu+2}^{m} p_{n-\nu} \frac{p_n}{P_n P_{n-1}} \right]$$

$$+ o\left[\sum_{\nu=0}^{m-2} \frac{\lambda_{\nu+1} P_{\nu+1} \log(\nu+1)}{(\nu+1)(\nu+2)\log^2(\nu+2)} \sum_{n=\nu+2}^{m} p_{n-\nu} \frac{p_n}{P_n P_{n-1}} \right]$$

$$= o\left[\sum_{\nu=0}^{m-2} \frac{\lambda_{\nu+1}}{\nu+1}\right] + o\left[\sum_{\nu=0}^{m-2} \frac{\Delta\lambda_{\nu+1}}{\nu+1}\right] + o\left[\sum_{\nu=0}^{m-2} \frac{\lambda_{\nu+1}}{(\nu+1)^2}\right]$$

$$= o(1) \qquad \ldots \quad (5.7)$$

as $m \to \infty$, since

$$P_n - P_{n-\nu-1} \leqslant (\nu+1)p_{n-\nu}$$

and

$$\sum_{n=\nu+2} \frac{p_{n-\nu}p_n}{P_n P_{n-1}} = o\left(\frac{1}{P_{\nu+1}}\right)$$

as $m \to \infty$.

Finally

$$\sum_{n=2}^{\infty} |J_2| = \sum_{n=2}^{\infty} \left| T_n \frac{p_n(P_n - P_0)\lambda_n}{n^2 P_{n-1} \log(n+1)} \right|$$

$$= \sum_{n=2}^{\infty} \left| \frac{p_n(P_n - P_0)\lambda_n}{n P_{n-1}} \cdot \frac{\log n}{\log(n+1)} \right| \qquad \ldots \quad (5.8)$$

$$= 0(1).$$

By virtue of (5.6)-(5.8) we get

$$\sum_{n=2}^{\infty} |J| = 0(1). \qquad \ldots \quad (5.9)$$

By (5.5) and (5.9) we get (5.1).

This completes the proof of the theorem.

Acknowledgement

The authors are thankful to the referee for his valuable suggestions.

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