ON ALMOST CONTACT MANIFOLDS WITH A SPECIFIED AFFINE CONNEXION

by R. D. S. Kushwaha, *Department of Mathematics, Banaras Hindu University, Varanasi 5*

(Communicated by F. C. Auluck, F.N.A.)

(*Received 3 July 1972*)

In the present paper, a specified affine connexion in an almost contact manifold is introduced and some of its properties are studied. Also, recurrent and birecurrent properties of the above manifold are studied considering its curvature tensor and projective tensor.

1. Introduction

Let us consider an $(n = 2m+1)$ dimensional real differentiable manifold V_n of differentiability class C^∞. Let there exist in V_n a C^∞ vector valued linear function F, a C^∞ vector field T and a C^∞ 1-form A satisfying

$$\bar{X} \overset{\text{def}}{=} F(X), A(T) = 1, A(\bar{X}) = 0, \bar{T} = 0, \bar{\bar{X}}+X = A(X)T \quad \ldots \quad (1.1)$$

for arbitrary vector field X. Then V_n is called an almost contact manifold and the structure (F, T, A) is called an almost contact structure.

Agreement 1.1—In what follows, the equations containing X, Y, Z, U, V will hold for arbitrary vector fields X, Y, Z, U, V.

Let D be an affine connexion in V_n and let S, a vector valued bilinear function, be its torsion tensor : $S(X, Y) = D_X Y - D_Y X - [X, Y]$, where [ ] denotes Lie brackets.

Agreement 1.2—The affine connexion D satisfies the following properties :

$$\left.\begin{aligned} &\text{(a)} \quad D_X T = \bar{X} \\ &\text{(b)} \quad D_X \bar{Y} = D_Y \bar{X} + \overline{[X, Y]} + XA(Y) - YA(X) \\ &\text{(c)} \quad (D_X A)(Y) + (D_Y A)(X) = 0 \text{ i.e. } A \text{ is a killng vector.} \end{aligned}\right\} \quad \ldots \quad (1.2)$$

Let $'F(X, Y) = (D_X A)(Y)$.

Then it has been shown (Mishra 1972)

$$\left.\begin{aligned} &(a)\quad 'F(X, Y) = -'F(Y, X) \\ &(b)\quad 'F(T, X) = 0 \\ &(c)\quad 'F(X, \overline{Y}) + 'F(\overline{X}, Y) = 0. \end{aligned}\right\} \qquad \ldots \quad (1.3)$$

Also in manifold V_n it has been shown (Mishra 1972)

$$\left.\begin{aligned} &(a)\quad (D_X A)(T) = 0 \\ &(b)\quad (D_T A)(X) = 0 \\ &(c)\quad A(D_T \overline{Y}) = 0. \end{aligned}\right\} \qquad \ldots \quad (1.4)$$

2. Affine Connexion

Theorem (2.1)—In manifold V_n

$$D_{\overline{Y}}\overline{X} - \overline{D_Y\overline{X}} + D_X Y + \overline{D_X\overline{Y}} = 2\overline{X}A(Y) - [X, Y] - \overline{[X, Y]} + TX(A(Y)) + TA([X, Y]) \qquad \ldots \quad (2.1)$$

$$\overline{D_X\overline{Y}} - \overline{D_Y\overline{X}} + \overline{D_{\overline{X}}Y} - \overline{D_{\overline{Y}}X} = [\overline{X}, \overline{Y}] - [X, Y] + TA([X, Y]) - TA([\overline{X}, \overline{Y}]). \qquad \ldots \quad (2.2)$$

Proof : Barring Y in (1.2*b*) and using (1.1) and (1.2*a*), we obtain

$$D_{\overline{Y}}\overline{X} + D_X Y = \overline{X}A(Y) + \overline{Y}A(X) - \overline{[X, \overline{Y}]} + TX(A(Y)). \qquad \ldots \quad (2.3a)$$

Also barring (1.2*b*) and (2.3*a*) and using (1.1), we obtain

$$\overline{D_X\overline{Y}} - \overline{D_Y\overline{X}} = \overline{X}A(Y) - \overline{Y}A(X) - [X, Y] + TA([X, Y]). \qquad \ldots \quad (2.3b)$$

Adding (2.3*a*) and (2.3*b*), we get (2.1). Again barring X in (2.3*a*) and barring the resulting equation we get

$$\overline{D_{\overline{X}}Y} - \overline{D_{\overline{Y}}X} = \overline{Y}A(X) - \overline{X}A(Y) + [\overline{X}, \overline{Y}] - TA([\overline{X}, \overline{Y}]). \qquad \ldots \quad (2.3c)$$

Barring (2.3*b*) and (2.3*c*), we at once obtain (2.2).

Theorem (2.2)—We have

$$D_T Y + \overline{D_T\overline{Y}} = TA(D_T Y) - [T, Y] - [T, \overline{Y}] + TA([T, Y]) \qquad \ldots \quad (2.4a)$$

$$\overline{D_T\overline{Y}} - \overline{D_{\overline{Y}}T} = TA([T, Y]) - [T, Y]. \qquad \ldots \quad (2.4b)$$

Proof : Putting T for X in (2.1) and using (1.1) and (1.4b), we obtain (2.4a). Putting T for X in (2.2) and using (1.1), we obtain (2.4b).

Theorem (2.3)—If S be the torsion tensor of D, then

$$\overline{(D_X F)(Y)}-\overline{(D_Y F)(X)}-S(Y, X)+TA(S(X, Y)) = \overline{D_{\bar{Y}} X}-\overline{D_{\bar{X}} Y}$$
$$+[\bar{X}, \bar{Y}]-TA([\bar{X}, \bar{Y}]) \quad \dots \quad (2.5a)$$

$$(D_X F)(Y)-(D_Y F)(X)+\overline{S(X, Y)} = TA((D_X F)(Y))+TA((D_Y F)(Y)). \quad \dots \quad (2.5b)$$

Proof : Using (1.1) in (2.3b), we obtain (2.5a). Barring (2.5a) and using (1.1), we obtain (2.5b).

Theorem (2.4)—Let N, a vector valued bilinear function, be the Nijenhuis tensor.

$$N(X, Y) \overset{\text{def}}{=} D_{\bar{X}} \bar{Y}-D_{\bar{Y}} \bar{\bar{X}}-D_X Y+D_Y X+TA(D_X Y-D_Y X)-$$
$$\overline{D_X \bar{Y}}+\overline{D_{\bar{Y}} X}-\overline{D_{\bar{X}} Y}+\overline{D_Y \bar{\bar{X}}}. \quad \dots \quad (2.6)$$

Then

$$N(T, Y) = -\overline{S(T, \bar{Y})} \quad \dots \quad (2.7a)$$

$$A(N(X, Y)) = 2'F(X, Y) \quad \dots \quad (2.7b)$$

$$N(\bar{\bar{X}}, Y)+N(X, \bar{Y}) = -A(X)\overline{S(T, Y)}-A(Y)S(X, T). \quad \dots \quad (2.7c)$$

Proof : From (2.3a), we have

$$D_{\bar{Y}} \bar{\bar{X}}-D_{\bar{X}} \bar{Y}+D_X Y-D_Y X = [Y, \bar{\bar{X}}]-[X, \bar{Y}]+T\{(D_X A)(Y)-(D_Y A)(X)$$
$$+A(D_X Y-D_Y X)\}.$$

Using the definition of $S(X, Y)$ and of ${}^1F(X, Y)$ in above, we get

$$N(X, Y)+\overline{S(\bar{\bar{X}}, Y)}+\overline{S(X, \bar{Y})}+2T'F(X, Y) = 0. \quad \dots \quad (2.8)$$

Putting T for X in (2.8) and using (1.3b) we get (2.7a). (2.7b) is also obtained by (2.8). Barring X and Y in (2.8) and adding the resulting equations and using (1.3c), we obtain (2.7c).

Theorem (2.5)—Let us put

$$P(X, Y) \overset{\text{def}}{=} D_{\bar{X}} \bar{Y}-D_{\bar{Y}} \bar{\bar{X}}-D_X Y+D_Y X+TA(D_X Y-D_Y X) \quad \dots \quad (2.9)$$

$$Q(X, Y) \overset{\text{def}}{=} D_{\bar{X}} \bar{Y}-D_{\bar{Y}} \bar{\bar{X}}-\overline{D_X \bar{Y}}+\overline{D_{\bar{Y}} X}+TA(D_X Y-D_Y X) \quad \dots \quad (2.10)$$

$$R(X, Y) \overset{\text{def}}{=} D_{\bar{X}} \bar{Y}-D_{\bar{Y}} \bar{\bar{X}}-\overline{D_{\bar{X}} Y}+\overline{D_Y \bar{\bar{X}}}+TA(D_X Y-D_Y X). \quad \dots \quad (2.11)$$

Then

$$A(P(X, Y)) = A(N(X, Y)) \qquad \dots (2.11a)$$

$$A(Q(X, Y)) = A(N(X, Y))+A(D_X Y)-A(D_Y X) \qquad \dots (2.11b)$$

$$A(R(X, Y)) = A(N(X, Y))+A(D_X Y)-A(D_Y X) \qquad \dots (2.11c)$$

$$A(Q(X, Y)) = A(R(X, Y)). \qquad \dots (2.11d)$$

Proof : Using (1.1) in (2.9), we get

$$A(P(X, Y)) = A(D_{\overline{X}}\overline{Y})-A(D_{\overline{Y}}\overline{X}).$$

Also using (1.1) in the expression for $N(X, Y)$, we get

$$A(N(X, Y)) = A(D_{\overline{X}}\overline{Y})-A(D_{\overline{Y}}\overline{X}).$$

From the above two equations, we at once get (2.11a). The proofs of (2.11b), (2.11c) are similar as that of (2.11a). (2.11d) is obvious.

Let us put

$$M(X, Y) \overset{\text{def}}{=} D_{\overline{X}}\overline{Y}-D_X Y-\overline{D_X\overline{Y}}-D_{\overline{X}}Y. \qquad \dots (2.12)$$

Theorem (2.6)—We have

$$M(X, Y) = N(X, Y)+2T'F(X, Y)+D_{\overline{X}}\overline{Y}-D_X Y-\overline{D_{\overline{Y}}X}-\overline{D_Y\overline{X}}+$$

$$\overline{[Y, \overline{X}]}-\overline{[X, \overline{Y}]} \qquad \dots (2.13a)$$

$$M(T, Y) = N(T, Y)-D_T Y-\overline{D_{\overline{Y}}T}-\overline{[T, \overline{Y}]} \qquad \dots (2.13b)$$

$$A(M(X, Y)) = A(N(T, Y))-A(D_T Y). \qquad \dots (2.13c)$$

Proof : From (2.3a), we have

$$D_{\overline{Y}}\overline{X}-D_{\overline{X}}\overline{Y}+D_X Y-D_Y X = \overline{[Y, \overline{X}]}-\overline{[X, \overline{Y}]}+$$

$$T\{(D_X A)(Y)-(D_Y A)(X)+A(D_X Y-D_Y X)\}.$$

Using the definitions of $N(X, Y)$ and of $'F(X, Y)$ and the equation (2.12), in above, we get (2.13a). Putting T for X in (2.13a), we obtain (2.13b). Using (1.1) in (2.13b), we get (2.13c).

Theorem (2.7)—Put

$$G(X, Y) \overset{\text{def}}{=} D_{\overline{X}}\overline{Y}-D_X Y. \qquad \dots (2.14)$$

Then

$$G(X, Y)+G(\overline{\overline{X}, Y}) = M(X, Y)+A(X)\overline{D_T\overline{Y}} \quad \dots (2.15a)$$

$$A(G(X, Y)) = A(M(X, Y)). \quad \dots (2.15b)$$

Proof : Barring X and then the whole equation and adding the resulting equation with (2.14) and using (2.12), we at once get (2.15a). Using (1.1) in (2.15a), we get (2.15b).

3. Curvature Tensor

Let K be the curvature tensor with respect to the connexion D :

$$K(X, Y, Z) \overset{\text{def}}{=} D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z. \quad \dots (3.1)$$

and Ric be the corresponding Ricci tensor.

$$\text{Ric}(Y, Z) \overset{\text{def}}{=} (C_1^1 K)(Y, Z). \quad \dots (3.2)$$

$K(X, Y, T)$ is obtained (Mishra 1972). Here $K(T, X, Y)$ is deduced.

Theorem (3.1)—We have

$$K(T, Y, Z) = TA(D_T D_Y Z) - [T, \overline{D_Y \overline{Z}}] + D_Y Z - \overline{Y}A(D_T Z) -$$

$$TD_Y(A(D_T Z)) + D_Y \overline{[T, \overline{Z}]} - D_Y \overline{Z} - D_{[T, Y]} Z \quad \dots (3.3a)$$

$$\text{Ric}(Y, Z) = (n-1)A(D_T D_Y Z) - \bar{\bar{D}}_Y Z - (n-1) D_Y(A(D_T Z)) -$$

$$D_Y \overline{Z} - \overline{Y}A(D_T Z). \quad \dots (3.3b)$$

Proof : From (3.1), we have

$$K(T, Y, Z) = D_T D_Y Z - D_Y D_T Z - D_{[T, Y]} Z.$$

Using

$$D_T Y = TA(D_T Y) - \overline{[T, \overline{Y}]} + \overline{Y} \quad \text{(Mishra 1972)} \quad \dots (3.3c)$$

in above equation and using (1.2a), we obtain (3.3a). Using (3.2) in (3.3a), we obtain (3.3b).

Theorem (3.2)—We have

$$K(T, Y, T) = TA(Y) - Y. \quad \dots (3.4)$$

Proof : We know

$$K(X, Y, T) = XA(Y) - YA(X).$$

Putting T for X in above we get (3.4).

Theorem (3.3)—The manifold V_n as defined above cannot be a flat manifold.

Proof :

$$K(X, Y, Z) = 0 \Longrightarrow K(T, Y, T) = 0 \Longrightarrow TA(Y) - Y = 0$$

which is impossible.

Theorem (3.4)—If the manifold V_n is a birecurrent manifold with B as the parameter of recurrence

$$K(Z, P, Y) = ZA(K(T, Y, P)) - PA(K(T, Y, Z)) + \{A(Y) + B(T, Y) - B(Y, T) + B([T, Y])\}\{ZA(P) - PA(Z)\} \quad \dots \quad (3.5)$$

$$\text{Ric}(P, Y) = (n-1)[A(K(T, Y, P)) + A(P)\{A(Y) + B(T, Y) - B(Y, T) + B([T, Y])\}] \quad \dots \quad (3.6)$$

$$L(Z, P) = (C_3^1 K)(Z, P) = A(K(T, Z, P)) - A(K(T, P, Z)) + A(P)\{B(T, Z) - B(Z, T) + B(T, Z) - B(Z, T) + B([T, Z]) - A(Z)\}B(T, P) - B(P, T) + B([T, P]). \quad \dots \quad (3.7)$$

Proof : Since the manifold is birecurrent

$$(D_X D_Y K)(Z, P, Q) = B(X, Y)K(Z, P, Q)$$

we get

$$(D_X D_Y K - D_Y D_X K - P_{[X, Y]} K)(Z, P, Q) = \{B(X, Y) - B(Y, X) + B([X, Y])\}K(Z, P, Q)$$

or

$$K(X, Y, K(Z, P, Q)) - K(K(X, Y, Z), P, Q) - K(Z, K(X, Y, P), Q) - K(Z, P, K(X, Y, Q)) = B(X, Y) - B(Y, X) + B([X, Y])K(Z, P, Q).$$

Putting T for X and Q in this equation and using

$$K(X, Y, T) = YA(Y) - YA(X) \quad \dots \quad (3.8)$$

we obtain (3.5). Using (3.2) we get (3.6). (3.7) follows from (3.5), the definition of L and (3.8).

Corollary—If the manifold is symmetric

$$K(Z, P, Y) = ZA(K(T, Y, P)) - PA(K(T, Y, Z)) + A(Y)\{ZA(P) - PA(Z)\}$$

$$\text{Ric}(P, Y) = (n-1)\{A(K(T, Y, P) + A(P)A(Y)\},$$

$$L(Z, P) = A(K(T, Z, P) - A(K(T, P, Z)).$$

Proof : Putting $B = 0$ in (3.5), (3.6) and (3.7) we obtain these relations.

4. Projective Tensor

Theorem (4.1)—Let W be the projective tensor with respect to the connection D :

$$W(X, Y, Z) \overset{\text{def}}{=} K(X, Y, Z)-\frac{1}{2m}[X \operatorname{Ric}(Y, Z)-Y \operatorname{Ric}(X, Z)] \quad \ldots \quad (4.1)$$

Then

$$W(X, Y, T)=0. \quad \ldots \quad (4.2)$$

Proof : From (4.1), we have

$$W(X, Y, T)=K(X, Y, T)-\frac{1}{2m}[X \operatorname{Ric}(Y, T)-Y \operatorname{Ric}(X, T)].$$

Using (3.8) and $R(Y, T)=(n-1)A(Y)$ in above.

$$W(X, Y, T)=XA(Y)-YA(X)-\frac{n-1}{2m}[XA(Y)-YA(X)]$$

which is (4.2).

Theorem (4.2)—The manifold V_n as defined above is always projectively flat.

Proof :

$$W(X, Y, Z)=0 \Longrightarrow W(X, Y, T)=0$$

which is always possible by (4.2).

Theorem (4.3)—We have

$$\begin{aligned} W(T, Y, Z) = {} & TA(D_T D_Y Z)-\overline{[T, D_Y \bar{Z}]}+\overline{D_Y \bar{Z}}-\bar{Y}A(D_T Z) \\ & -TD_Y(A(D_T Z))+D_Y\overline{[T, \bar{Z}]}-D_Y\bar{Z}-D_{[T, Y]}Z- \\ & \frac{1}{2m}[T \operatorname{Ric}(Y, Z)-YA(Z)]. \quad \ldots \quad (4.3a) \end{aligned}$$

$$\begin{aligned} A(W(T, Y, Z)) = {} & A(D_T D_Y Z)-D_Y(A(D_T Z))-A(D_{[T,Y]} Z)- \\ & \frac{1}{2m}[\operatorname{Ric}(Y, Z)-A(Y)A(Z)]+A(D_Y F)[T, \bar{Z}]-A(D_Y F)Z. \quad \ldots \quad (4.3b) \end{aligned}$$

Proof : Using (4.1), (3.3c) and (1.2a), we obtain (4.3a). Using (1.1) and (3.3a) in (4.3a), we get (4.3b).

Theorem (4.4)—If the manifold V_n is projectively recurrent manifold with B as the parameter of recurrence

$$\begin{aligned} & W(T, Y, Z)A(P)-PA(W(T, Y, Z))-W(T, Y, P)A(Z)+ZA(W(T, Z, P)) \\ & +\{(D_T B)(Y)-(D_Y B)(T)+B(S(T, Y))\}W(Z, P, T). \quad \ldots \quad (4.4) \end{aligned}$$

Proof : Since the manifold is projectively recurrent

$$(D_Y W)(Z, P, Q) = B(Y) W(Z, P, Q)$$

we get

$$K(X, Y, W(Z, P, Q)) - K(W(X, Y, Z)), P, Q) - K(Z, W(X, Y, P), Q) -$$
$$K(Z, P, W(X, Y, Q)) = \{(D_X B)(Y) - (D_Y B)(X) + B(S(X, Y))\} K(Z, P, W).$$

Putting T for X and for Q in this equation and using (3.8), we obtain (4.4).

Corollary (4.1)—If the manifold as defined above is a symmetric manifold.

$$W(T, Y, Z)A(P) - PA(W(T, Y, Z)) + W(T, Y, P)A(Z) -$$
$$ZA(W(T, Y, P)) = 0. \quad \ldots (4.5)$$

Proof : Putting $B = 0$ in (4.4), we obtain (4.5).

Acknowledgement

The author's thanks are due to Prof. R. S. Mishra for helpful suggestions.

Bibliography

Mishra, R. S. (1965). A Course in Tensor with Application to Riemannian Geometry. Pothishala Private Limited, Allahabad, pp. 16–81.

——— (1972). Almost contact manifolds with a specified affine connexion. *Tensor, N.S.*, **23**, 41–45.

Takano, K. (1971). On a space with bi-recurrent curvature. *Tensor, N.S.*, **22**, 329–32.