TORSION OF A RADIALLY COMPOSITE CIRCULAR CYLINDRICAL BEAM BY A GREEN'S FUNCTION TECHNIQUE FOR A COMPOSITE BODY

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(Communicated by F. C. Auluck, F.N.A.)

(*Received 2 May 1972*)

Solutions of the torsion problems of circular cylindrical beams, solid and hollow, composed of isotropic materials are obtained by a potential method involving use of a Green's identity and Green's function. The problems are of Saint Venant type in classical elasticity. The results obtained in the solid case agree with those stated by Muskhelishvili (1963) who followed a complex variable method.

1. INTRODUCTION

The axis of the twist which is the z-axis of the chosen cylindrical (r, θ, z) coordinate system is coincident with the geometric axis of symmetry of the circular cylinder. For the torision problem the three displacement components are given by $u_r = 0$, $u_\theta = \alpha z r$, $u_z = \alpha\phi(r, \theta)$, where α is the angle of twist per unit length of the beam and $\phi(r, \theta)$ the torsion function. The components of strain are then expressed as

$$e_{rr} = 0, \qquad e_{\theta\theta} = 0, \qquad e_{zz} = 0, \qquad e_{r\theta} = 0,$$

$$e_{zr} = \frac{\alpha}{2}\frac{\partial\phi}{\partial r}, \qquad e_{\theta z} = \tfrac{1}{2}\,\alpha\left(r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

while the stress components are

$$\tau_{rr} = 0, \qquad \tau_{\theta\theta} = 0, \qquad \tau_{zz} = 0, \qquad \tau_{r\theta} = 0,$$

$$\tau_{zr} = \alpha\mu\frac{\partial\phi}{\partial r}, \qquad \tau_{\theta z} = \alpha\mu\left(r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right).$$

Introducing the Prandtl Stress function $P(r, \theta)$ defined by

$$\frac{\partial P}{\partial\theta} = r\frac{\partial\phi}{\partial r}, \qquad \frac{\partial P}{\partial r} = -\left(r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

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we have the last two shear stress components given by

$$\tau_{zr} = \frac{\alpha\mu}{r} \cdot \frac{\partial P}{\partial \theta}, \qquad \tau_{\theta z} = -\alpha\mu \frac{\partial P}{\partial r}. \qquad \ldots (1.1)$$

It is readily checked that P satisfies

$$\nabla^2 P = -2 \qquad \ldots (1.2)$$

where ∇^2 is the two-dimensional Laplacian

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

2. Case of a Solid Beam

Let the cross-section of the beam be given by $0_+ \leqslant r \leqslant b_-$, $0_+ \leqslant \theta \leqslant 2\pi_-$, where $r = 0_+$ means 'r approaches 0 through values greater than 0' and $r = b_-$, means 'r approaches b through values less than b', etc. First the beam is taken to be formed of two isotropic media $S_1 : 0_+ \leqslant r \leqslant a_-$, $0_+ \leqslant \theta \leqslant 2\pi_-$; $S_2 : a_+ \leqslant r \leqslant b_-$, $0_+ \leqslant \theta \leqslant 2\pi_-$. Let μ_1, μ_2 be the moduli of rigidity of the materials forming S_1, S_2. The entire torsion problem can be reduced to the problem of finding the Prandtl stress functions P_1, P_2 in S_1, S_2 respectively. The interface is given by $r = a$.

Following Muskhelishvili (1963), we have the boundary condition

$$P_1|_{r=b_-} = 0 \qquad \ldots (2.1)$$

and the continuity conditions

$$P_1|_{r=a_-} = kP_2|_{r=a_+}, \quad \text{where } k = \mu_2/\mu_1, \qquad \ldots (2.2)$$

$$\left.\frac{\partial P_1}{\partial r}\right|_{r=a_-} = \left.\frac{\partial P_2}{\partial r}\right|_{r=a_+}. \qquad \ldots (2.3)$$

(2.3) results from the continuity of the z-displacement component u_z across $r = a$ while (2.2) results from the continuity of τ_{zr} across $r = a$. In deducing (2.2) and (2.3) the non-essential arbitrary constants are dropped (*see* Muskhelishvili 1963, Sokolnikoff 1956). Now we proceed to find the Green's function $G_1(r, \theta | r_0, \theta_0)$ which satisfies

$$\nabla^2 G_1 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G_1}{\partial r}\right)+\frac{1}{r^2}\cdot\frac{\partial^2 G_1}{\partial\theta^2}$$

$$= -\frac{4\pi}{r}\delta(r-r_0)\delta(\theta-\theta_0) \text{ in } S_1 \qquad \ldots (2.4)$$

with

$$G_1|_{r=a_-} = 0.$$

We have to make a preliminary change of dependent variable to avoid convergence difficulty at $r = 0$. Let $G_1 = G_0+H$, where G_0 is the value of G_1 at the origin. Using this in (2.4)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 H}{\partial \theta^2} = -\frac{4\pi}{r}\delta(r-r_0)\delta(\theta-\theta_0). \quad \ldots (2.5)$$

Noticing $G_1|_{r=b_-} = 0$ we apply the Mellin sine transform (*see* Stakgold 1968), viz. multiply (2.5) throughout by $S_t(r)$ where $S(r) = \sin(t \log_e r/a)$, and integrate with respect to r between $r = 0$ and $r = a$. This gives on simplification

$$\frac{d^2\overline{H}}{d\theta^2}+tG_0-t^2\overline{H} = -4\pi\delta(\theta-\theta_0)S_t(r_0) \quad \ldots (2.6)$$

where

$$\overline{H} = \int_0^a \frac{1}{r} S_t(r)H dr \quad \ldots (2.7)$$

so that

$$H = \frac{2}{\pi}\int_0^\infty \overline{H}S_t(r)dt. \quad \ldots (2.8)$$

Let

$$\overline{H} = \frac{G_0}{t}+4\pi\overline{K}. \quad \ldots (2.9)$$

Then

$$G_1 = G_0+H = G_0+\frac{2}{\pi}\int_0^\infty \left(\frac{G_0}{t}+4\pi\overline{K}\right) S_t(r)dt$$

or, noticing that

$$\int_0^\infty \frac{1}{t} S_t(r)dt = -\frac{\pi}{2}$$

$$G_1 = 8\int_0^\infty \overline{K}S_t(r)dt. \quad \ldots (2.10)$$

Now using (2.9) in (2.6).

$$\frac{d^2\overline{K}}{d\theta^2}-t^2\overline{K} = -\delta(\theta-\theta_0)S_t(r_0). \quad \ldots (2.11)$$

Let

$$\overline{K} = A_0(t)+\sum_{n=1}^\infty A_n(t)\cos n\theta+\sum_{n=1}^\infty B_n(t)\sin n\theta. \quad \ldots (2.12)$$

Using this in (2.11)

$$t^2A_0(t)+\sum_{n=1}^{\infty}(t^2+n^2)\{A_n(t)\cos n\theta+B_n(t)\sin n\theta\}=\delta(\theta-\theta_0)S_t(r_0). \quad \ldots (2.13)$$

Integrating (2.13) with respect to θ between $\theta=0$ and $\theta=2\pi$, we get

$$A_0=\frac{S_t(r_0)}{2\pi t^2}.$$

Multiplying (2.13) by $\cos n\theta$ and integrating with respect to θ between $\theta=0$ and $\theta=2\pi$ we obtain

$$A_n=\frac{S_t(r_0)\cos n\theta_0}{\pi(t^2+n^2)}$$

and in a similar way multiplying (2.13) by $\sin n\theta$ and integrating with respect to θ between $\theta=0$ and $\theta=2\pi$ we get

$$B_n=\frac{S_t(r_0)\sin n\theta}{\pi(t^2+n^2)}.$$

Substituting all these in (2.12) we get

$$\overline{K}=\frac{S_t(r)}{2\pi t^2}+\sum_{n=1}^{\infty}\frac{S_t(r_0)}{\pi(t^2+n^2)}\cos n(\theta-\theta_0)$$

and hence from (2.10) we have

$$G_1=\frac{4}{\pi}\int_0^{\infty}\frac{S_t(r)S_t(r_0)}{t^2}\,dt+\frac{8}{\pi}\sum_{n=1}^{\infty}\int_0^{\infty}\frac{S_t(r)S_t(r_0)}{t^2+n^2}\cos n(\theta-\theta_0)dt \quad \ldots (2.14)$$

We proceed now to find the Green's function $G_2(r,\theta\,|\,r_0,\theta_0)$ which satisfies

$$\nabla^2G_2=-\frac{4\pi}{r}\delta(r-r_0)\delta(\theta-\theta_0) \text{ in } S_2 \quad \ldots (2.15)$$

with

$$G_2\big|_{r=a_+}=0, \qquad G_2\big|_{r=b_-}=0.$$

Let

$$G_2=g_0(r)+\sum_{n=1}^{\infty}g_n(r)\cos n\theta+\sum_{n=1}^{\infty}h_n(r)\sin n\theta. \quad \ldots (2.16)$$

Then, from (2.15) we obtain

$$\frac{d}{dr}\left(r\frac{dg_0}{dr}\right)+\sum_{n=1}^{\infty}\left\{\frac{d}{dr}\left(r\frac{dg_n}{dr}\right)-n^2g_n\right\}\cos n\theta$$

$$+\sum_{n=1}^{\infty}\left\{\frac{d}{dr}\left(r\frac{dh_n}{dr}\right)-n^2h_n\right\}\sin n\theta=-4\pi\,\delta(r-r_0)\,\delta(\theta-\theta_0) \quad \ldots (2.17)$$

Integrating (2.17) with respect to θ between $\theta = 0$ and $\theta = 2\pi$ we get

$$\frac{d}{dr}\left(r\frac{dg_0}{dr}\right) = -2\delta(r-r_0). \qquad \ldots (2.18)$$

The corresponding homogeneous form is

$$\frac{d}{dr}\left(r\frac{dg_0}{dr}\right) = 0. \qquad \ldots (2.19)$$

The solutions of (2.19) satisfying $g_0|_{r=a_+} = 0$, $g_0|_{r=b_-} = 0$ are $\log_e r/a$ and $\log_e b/r$.

Let

$$g_0 = D_0 p_0(r_>) q_0(r_<) \qquad \ldots (2.20)$$

where

$$p_0(r) = \log_e \frac{b}{r}, \quad q_0(r) = \log_e \frac{r}{a}, \quad r_> = \max(r, r_0), \quad r_< = \min(r, r_0).$$

Integrating (2.18) with respect to r between $r = a$ and $r = b$, and introducing (2.20) we get

$$D_0 = \frac{2}{\log_e \frac{b}{a}}.$$

Therefore,

$$g_0(r) = \frac{2}{\log_e \frac{b}{a}} p_0(r_>)\, q_0(r_<).$$

Multiplying (2.17) by $\cos n\theta$ and integrating with respect to θ between $\theta = 0$ and $\theta = 2\pi$ we obtain

$$\frac{d}{dr}\left(r\frac{dg_n}{dr}\right) - n^2 g_n = -4\cos n\theta_0\, \delta(r-r_0). \qquad \ldots (2.21)$$

The solutions of the corresponding homogeneous equation which satisfy

$g_n|_{r=b_-} = 0$, $g_n|_{r=a_+} = 0$ are $p_n(r) = \sinh(n \log_e b/r)$, $q_n(r) = \sinh(n \log_e r/a)$, $n = 1, 2, 3, \ldots$.

Let

$$g_n = D_n p_n(r_>) q_n(r_<). \qquad \ldots (2.22)$$

Using this in (2.21), after integration with respect to r between $r = a$ and $r = b$ we obtain on simplification

$$D_n = \frac{4\cos n\theta_0}{n \sinh\left(n \log_e \frac{b}{a}\right)}$$

Therefore,

$$g_n(r) = \frac{4\cos n\theta \cos n\theta_0}{n \sinh\left(n \log_e \frac{b}{a}\right)} p_n(r_>) q_n(r_<).$$

Similarly

$$h_n(r) = \frac{4\sin n\theta \sin n\theta_0}{n \sinh\left(n \log_e \frac{b}{a}\right)} p_n(r_>) q_n(r_<).$$

Hence, from (2.16)

$$G_2 = \frac{2}{\log_e \frac{b}{a}} p_0(r_>) q_0(r_<) + 4 \sum_{n=1}^{\infty} \frac{\cos n(\theta-\theta_0)}{n \sinh\left(n \log_e \frac{b}{a}\right)} p_n(r_>) q_n(r_<) \quad \ldots (2.23)$$

By a Green's identity in a plane

$$\iint_S (P\nabla^2 G - G\nabla^2 P) r_0 dr_0 d\theta_0 = \sum_{n=1}^{s} \int_{L_p} \left(P \frac{\partial G}{\partial n_p} - G \frac{\partial P}{\partial n_p}\right) dL_p \quad \ldots (2.24)$$

where L_p's are the boundaries of the closed simply-connected cross-section S, δn_p a differential element of the outer drawn normal to L_p at a source point on it. In (2.24) the surface integral is to be taken over all the source points (r_0, θ_0) in S while the line integrals are to be taken over all the source points (r_0, θ_0) on L_p, $p = 1, 2, \ldots, s$.

Applying (2.24) to the regions S_1, S_2 with G_1, G_2 given by (2.14) and (2.23)

$$\iint_{S_1} (P_1\nabla^2 G_1 - G_1\nabla^2 P_1) r_0 dr_0 d\theta_0 = \int_0^{2\pi} P_1 \left|_{r_0=a_-} \frac{\partial G_1}{\partial r_0}\right|_{r_0=a_+} a\, d\theta_0 \quad \ldots (2.25)$$

$$\iint_{S_2} (P_2\nabla^2 G_2 - G_2\nabla^2 P_2) r_0 dr_0 d\theta_0 = -\int_0^{2\pi} P_2 \left|_{r_0=a_+} \frac{\partial G_2}{\partial r_0}\right|_{r_0=a_+} a\, d\theta_0 \quad \ldots (2.26)$$

Now, by symmetry, P_1 and P_2 are functions of r only. Let $P_2|_{r_0=a_+} = A$ (constant), then by (2.2)

$$P_1|_{r_0=a_-} = kA.$$

By counter integration in a complex plane

$$\int_0^{\infty} \frac{1}{t} \sin\left(t \log_e \frac{r}{a}\right) dt = -\frac{\pi}{2}, \quad \int_0^{\infty} \frac{t \sin\left(t \log_e \frac{r}{a}\right)}{t^2+n^2} dt = -\frac{\pi}{2}\left(\frac{r}{a}\right)^n.$$

Using these we get

$$\left.\frac{\partial G_1}{\partial r_0}\right|_{r_0=a_-} = -\frac{2}{a}-\frac{4}{a}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\theta_0).$$

Also, by contour integration in a complex plane we obtain

$$\int_0^\infty \frac{\sin\left(t\log_e\frac{r}{a}\right)}{t(t^2+4)}\,dt = \frac{\pi}{8}\left(\frac{r^2}{a^2}-1\right), \quad \int_0^\infty \frac{\sin\left(t\log_e\frac{r}{a}\right)}{(t^2+n^2)(t^2+4)}\,dt$$

$$= \frac{\pi}{2(n^2-4)}\left\{\left(\frac{r}{a}\right)^n-\frac{r^2}{a^2}\right\} \qquad \text{for } n \neq 2.$$

Using all these one obtains from (2.25)

$$P_1 = \frac{a^2-r^2}{2}+kA$$

and from (2.26)

$$P_2 = -\frac{r^2}{2}+\frac{1}{2\log_e b/a}\{(2A+a^2)\log_e b/r+b^2\log_e r/a\}.$$

Applying then the other continuity condition (2.3) A is obtained as

$$A = \tfrac{1}{2}(b^2-a^2).$$

Therefore,

$$P_1 = \tfrac{1}{2}\{(a^2-r^2)+k(b^2-a^2)\}, \qquad \ldots (2.27)$$

and

$$P_2 = \tfrac{1}{2}(b^2-r^2). \qquad \ldots (2.28)$$

The shear-stress components τ_{zr}, $\tau_{\theta z}$ in S_1, S_2 can be obtained by (1.1). The torsional rigidity D of the beam is given by

$$D = 2\mu_1\int_{r=0}^{a}\int_{\theta=0}^{2\pi} P_1\,r\,d\theta\,dr+2\mu_2\int_{r=a}^{b}\int_{\theta=0}^{2\pi} P_2\,r\,d\theta\,dr$$

which on the introductions of (2.27), (2.28) leads to

$$D = \tfrac{1}{2}\mu_2\pi b^4+\tfrac{1}{2}(\mu_1-\mu_2)\pi a^4.$$

This agrees with the result given in Muskhelishvili (1963) (*see* page 621, where l is to be put equal to zero). The above procedure can be adopted for a composition of any number of such constituent isotropic media,

3. Case of a Hollow Beam

Let the cross-section of the beam be given by $a_+ \leqslant r \leqslant c_-$, $0_+ \leqslant \theta \leqslant 2\pi$. First the beam is taken to be formed of two isotropic media $S_1 : a_+ \leqslant r \leqslant b_-$, $0_+ \leqslant \theta \leqslant 2\pi_-$; $S_2 : b_+ \leqslant r \leqslant c_-$, $0_+ \leqslant \theta \leqslant 2\pi_-$. Let μ_1, μ_2 be the moduli of rigidity of the materials forming S_1, S_2. Let P_1 and P_2 be the values of the Prandtl stress function P in S_1, S_2 respectively. Then $\nabla^2 P_1 = -2$ in S_1, $\nabla^2 P_2 = -2$ in S_2. In the manner (2.23) is obtained the Green's function $G_1(r, \theta \mid r_0, \theta_0)$ which satisfies

$$\nabla^2 G_1 = -\frac{4\pi}{r}\,\delta(r-r_0)\,\delta(\theta-\theta_0)$$

with

$$G_1|_{r=a_+} = 0, \qquad G_1|_{r=b_-} = 0$$

can be deduced as

$$G_1 = \frac{2}{\log_e a/b}\,p_0(r_>)\,q_0(r_<) + 4\sum_{n=1}^{\infty}\frac{\cos n(\theta-\theta_0)}{n\sinh(n\log_e a/b)}\,p_n(r_>)q_n(r_<) \quad \ldots \quad (3.1)$$

and $G_2(r, \theta \mid r_0, \theta_0)$ which satisfies

$$\nabla^2 G_2 = -\frac{4\pi}{r}\,\delta(r-r_0)\,\delta(\theta-\theta_0)$$

in S_2 with

$$G_2|_{r=b_+} = 0, \qquad G_2|_{r=c_-} = 0 \text{ as}$$

$$G_2 = \frac{2}{\log_e c/b}\,a_0(r_>)\,b_0(r_<) + 4\sum_{n=1}^{\infty}\frac{\cos n(\theta-\theta_0)}{n\sinh(n\log_e c/b)}\,a_n(r_>)b_n(r_<) \quad \ldots \quad (3.2)$$

where

$$a_0(r) = \log_e c/r, \quad b_0(r) = \log_e r/b, \quad a_n(r) = \sinh(n\log_e c/r),$$
$$b_n(r) = \sinh(n\log_e r/b), \qquad n = 1, 2, 3, \ldots.$$

Since the cross-section is a multiply-connected region the boundary conditions are

$$P_2|_{r=c_-} = 0, \qquad P_1|_{r=a_+} = \lambda \quad \ldots \quad (3.3)$$

where λ is an unknown non-zero constant (*see* Sokolnikoff 1956). In this case also, by symmetry, P_1 and P_2 are functions of r only. The continuity conditions are

$$P_2|_{r=b_+} = kA, \text{ where } k = \mu_1/\mu_2,\ A(\text{constant}) = P_1|_{r=b_-} \quad \ldots \quad (3.4)$$

and

$$\left.\frac{dP_1}{dr}\right|_{r=b_-} = \left.\frac{dP_2}{dr}\right|_{r=b_+}. \qquad \ldots \quad (3.5)$$

Applying (2.24) to each of S_1, S_2 and using (3.1) to (3.4) we finally obtain

$$P_1 = -\frac{r^2}{2} + \frac{1}{2\log_e b/a}\left(2\lambda \log_e \frac{b}{r} + 2A \log_e \frac{r}{a} + a^2 \log_e \frac{b}{r} + b^2 \log_e \frac{r}{a}\right) \quad (3.6)$$

$$P_2 = -\frac{r^2}{2} + \frac{1}{2\log_e c/b}\left(2kA \log_e \frac{c}{r} + b^2 \log_e \frac{c}{r} + c^2 \log_e \frac{r}{b}\right). \qquad \ldots \quad (3.7)$$

Now following Sokolnikoff (1956) the condition for single-valuedness referred to a multiply-connected region is

$$\int_0^{2\pi} \left.\frac{dP_1}{dr}\right|_{r=a} a\, d\theta = -2\pi a^2. \qquad \ldots \quad (3.8)$$

On the introduction of (3.6) in (3.8) we obtain on simplification

$$\lambda = A + \tfrac{1}{2}(b^2 - a^2). \qquad \ldots \quad (3.9)$$

Then using (3.6), (3.7) and (3.9) in the continuity condition (3.5) one obtains

$$A = \frac{c^2 - b^2}{2k}. \qquad \ldots \quad (3.10)$$

So from (3.9)

$$\lambda = \frac{1}{2k}\{c^2 - b^2 + k(b^2 - a^2)\}.$$

Accordingly, from (3.6), (3.7)

$$P_1 = \frac{1}{2k}\{k(b^2 - r^2) + c^2 - b^2\}, \qquad \ldots \quad (3.11)$$

$$P_2 = \tfrac{1}{2}(c^2 - r^2). \qquad \ldots \quad (3.12)$$

The stress components τ_{zr}, $\tau_{\theta z}$ in S_1, S_2 can be obtained by (1.1). Following Sokolnikoff (1956) the torsional rigidity D of the beam is given by

$$D = 2\mu_1 \int_{r=a}^{b} \int_{\theta=0}^{2\pi} P_1\, r d\theta\, dr + 2\mu_2 \int_{r=b}^{c} \int_{\theta=0}^{2\pi} P_2\, r d\theta\, dr + 2\mu_1 \lambda \pi a^2.$$

which on the introductions of (3.11) and (3.12) gives

$$D = \tfrac{1}{2}\pi\mu_1(b^4 - a^4) + \tfrac{1}{2}\pi\mu_2(c^4 - b^4). \qquad \ldots \quad (3.13)$$

In particular, when $\mu_2 = 0$, $D = \frac{1}{2}\pi\mu_1(b^4 - a^4)$ which agrees with the result for a hollow circular cylindrical beam given in Sokolnikoff (1956). This procedure also can be adopted for a composition of any number of such constituent isotropic media in a similar way.

For a thin tube reinforced by a uniformly thin coating given by $b = a(1+\epsilon_1)$, $c = b(1+\epsilon_2)$, where $0 < \epsilon_1, \epsilon_2 << 1$, retaining terms linear in ϵ_1, ϵ_2 only we have form (3.13), D is approximately equal to

$$2\pi a^4 \sum_{j=1}^{2} \mu_j \epsilon_j.$$

In the general case D is approxmately equal to

$$2\pi a^4 \sum_{j=1}^{n} \mu_j \epsilon_j.$$

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