ON ALMOST CONTACT MANIFOLDS : NIJENHUIS TENSOR

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In this paper the author expresses the Nijenhuis tensor in various forms in almost contact manifolds.

Consider an $n(=2m+1)$ dimensional real differentiable manifold V_n. Let there exist a C^∞ tensor field F of type (1, 1), a vector field T and a C^∞1-form A, satisfying

$$\bar{\bar{X}} = A(X)T - X, \text{ for an arbitrary vector field } X \qquad \dots \quad (1.1a)$$

where

$$\bar{X} \overset{\text{def}}{=} F(X) \qquad \dots \quad (1.1b)$$

$$A(T) = 1 \qquad \dots \quad (1.2)$$

$$\bar{T} = 0 \qquad \dots \quad (1.3)$$

$$A(\bar{X}) = 0. \qquad \dots \quad (1.4)$$

Then the manifold is said to have an almost contact structure (F, T, A) and V_n is said to be an almost contact manifold (Sasaki and Hatakeyama 1962).

Let the almost contact manifold V_n be endowed with the non-singular metric tensor g. If the following equation holds

$$g(\bar{X}, \bar{Y}) = g(X, Y) - A(X)\,A(Y) \qquad \dots \quad (1.5)$$

the manifold V_n is called an almost contact metric manifold or an almost Grayan manifold

Let us put

$$'F(X, Y) \overset{\text{def}}{=} g(\bar{X}, Y). \qquad \dots \quad (1.6)$$

Then, putting T for X in (1.5) and using (1.3), we get

$$g(T, Y) = A(Y). \qquad \dots \quad (1.7)$$

From (1.7), (1.6) and (1.1a), we have

$$'F(X, \bar{Y}) = g(\bar{X}, \bar{Y}) = g(X, Y) - A(X)A(Y) = -'F(\bar{X}, Y). \qquad \dots \quad (1.8)$$

Putting T for X in (1.7) and using (1.3), we get

$$'F(T, Y) = 0. \qquad \ldots \quad (1.9)$$

Barring X and Y in (1.8), and using (1.1), (1.4) and (1.9), we get

$$'F(\bar{X}, \bar{Y}) = 'F(X, Y) = g(\bar{X}, Y) = -g(X, \bar{Y}). \qquad \ldots \quad (1.10)$$

The equation (1.10) shows that $'F$ is hybrid in both the solts. Since g is symmetric, we have in consequence of (1.6) and (1.10)

$$'F(X, Y) = -g(X, \bar{Y}) = -g(\bar{Y}, X) = -'F(Y, X) \qquad \ldots \quad (1.11)$$

that is $'F$ is skew symmetric.

Finally using (1.3) in (1.7), we obtain

$$g(T, \bar{Y}) = 0. \qquad \ldots \quad (1.12)$$

Nijenhuis tensor of F in an almost contact metric (Grayan) manifold is a vector valued bilinear scalar function N, given by

$$N(X, Y) \overset{\text{def}}{=} [\bar{X}, \bar{Y}] + [\overline{\overline{X, Y}}) - [\overline{X, \bar{Y}}] - [\overline{\bar{X}, Y}]. \qquad \ldots \quad (1.13)$$

Using (1.1a), we have in an almost Grayan manifold

$$N(X, Y) = [\bar{X}, \bar{Y}] - [X, Y] - [\overline{X, \bar{Y}}] - [\overline{\bar{X}, Y}] + A([X, Y])T. \qquad \ldots \quad (1.14)$$

Almost Grayan manifold is called Grayan manifold if Nijenhuis tensor vanishes.

Theorem (1.1)—Let us put

$$P(X, Y) = [\bar{X}, \bar{Y}] - [\overline{X, \bar{Y}}]. \qquad \ldots \quad (1.15)$$

Then

$$P(X, Y) + \overline{P(X, \bar{Y})} = N(X, Y) + A([X, Y])T - A(Y)A([X, T])T$$
$$+ A(Y)[X, T] + A(Y)[\overline{\bar{X}, T}]. \qquad \ldots \quad (1.16)$$

Proof : Barring Y in (1.15) and using (1.1a), we obtain

$$P(X, \bar{Y}) = -[\bar{X}, Y] + A(Y)[\bar{X}, T] + [\overline{X, Y}] - \overline{A(Y)[X, T]}.$$

Barring the whole equation and using (1.1a), we get

$$\overline{P(X, \bar{Y})} = -[\overline{\bar{X}, Y}] - [X, Y] + A([X, Y])T + \overline{A(Y)[\bar{X}, T]} + A(Y)[X, T]$$
$$-A(Y)A([X, T])T.$$

Adding this resulting equation with (1.15) and using (1.14), we get

$$P(X, Y) + \overline{P(X, \bar{Y})} = N(X, Y) - A(Y)A([X, T])T + A(Y)[X, T] + \overline{A(Y)[\bar{X}, T]}$$

which is (1.16).

Corollary (1.1)—In Grayan manifold, we have

$$N(X, T) = P(X, T) - [X, T] - \overline{[\overline{X}, T]} + A([X, T])T. \quad \ldots \quad (1.17)$$

Proof : Putting T for Y in (1.16) and using (1.3), we get (1.17).

Theorem (1.2)—Let us put

$$Q(X, Y) = [\overline{X}, \overline{Y}] - \overline{[\overline{X}, Y]}. \quad \ldots \quad (1.18)$$

Then

$$Q(X, Y) + \overline{Q(\overline{X}, Y)} = N(X, Y) + A(X)[T, Y] + \overline{A(X)[T, \overline{Y}]}$$
$$- A(X)A([T, Y])T. \quad \ldots \quad (1.19)$$

Proof : Barring X in (1.18) and using (1.1a), we obtain

$$Q(\overline{X}, Y) = -[X, \overline{Y}] + A(X)[T, \overline{Y}] + \overline{[X, Y)} - \overline{A(X)[T, Y]}.$$

Barring the whole equation and using (1.1a), we get

$$\overline{Q(\overline{X}, Y)} = -\overline{[X, \overline{Y}]} + \overline{A(X)[T, \overline{Y}]} - [X, Y] + A([X, Y])T$$
$$+ A(X)[T, Y] - A(X)\ A([T, Y])T.$$

Adding this resulting equation to (1.18) and using (1.14), we get

$$Q(X, Y) + \overline{Q(\overline{X}, Y)} = N(X, Y) + A(X)[T, Y] + \overline{A(X)[T, \overline{Y}]} - A(X)A([T, Y])T$$

which is (1.19).

Corollary (1.2)—In almost Grayan manifold, we have

$$N(T, Y) = Q(T, Y) - \overline{[T, \overline{Y}]} - [T, Y] + A([T, Y])T. \quad \ldots \quad (1.20)$$

Proof : Putting T for X in (1.19) and using (1.3) we get (1.20).

Theorem (1.3)—Let us put

$$H(X, Y) = [\overline{X}, \overline{Y}] - [X, Y]. \quad \ldots \quad (1.21)$$

Then

$$H(X, Y) + \overline{H(\overline{X}, Y)} = N(X, Y) - A([X, Y])T + \overline{A(X)[T, \overline{Y}]} \quad \ldots \quad (1.22)$$

Proof : Barring X in (1.21) and using (1.1a), we obtain

$$H(\overline{X}, Y) = -[X, \overline{Y}] + A(X)[T, \overline{Y}] - [\overline{X}, Y].$$

Barring the whole equation and using (1.1a), we get

$$\overline{H(\overline{X}, Y)} = -\overline{[X, \overline{Y}]} + \overline{A(X)[T, \overline{Y}]} - \overline{[\overline{X}, Y]}.$$

Adding this equation to (1.21) and using (1.14), we get

$$\overline{H(\overline{X}, Y)}+H(X, Y) = N(X, Y)-A([X, Y]T+\overline{A(X)[T, \overline{Y}]}$$

which is (1.22).

Corollary (1.3)—The equation (1.22) is equivalent to

$$N(T, Y) = H(T, Y)+A([T, Y])T+[\overline{T, \overline{Y}}]. \quad \ldots \quad (1.23)$$

Proof : Putting T for X in (1.22) and using (1.3), we get (1.23).

Theorem (1.4)—In almost Grayan manifold

$$Q(T, Y)-H(T, Y) = [T, Y]-2A([T, Y])T. \quad \ldots \quad (1.24)$$

Proof : From (1.20) and (1.23) by equating the right-hand sides, we at once get (1.24).

Theorem (1.5)—In almost Grayan manifold, we have

$$\begin{aligned} P(X, Y)+\overline{P(X, \overline{Y})} &= Q(X, Y)+\overline{Q(\overline{X}, Y)}+A([X, Y])T \\ &\quad -A(Y)A([X, T])T+A(Y)[X, T]+\overline{A(Y)[\overline{X}, T]} \ldots \\ &\quad -A(X)[T, Y]-\overline{A(X)[T, \overline{Y}]}+A(X)A([T, Y])T. \quad \ldots \quad (1.25) \end{aligned}$$

Proof : From (1.16) and (1.19), we at once get (1.25).

Theorem (1.6)—In almost Grayan manifold, we have

$$\begin{aligned} P(X, Y)+P(\overline{X, \overline{Y}}) &= H(X, Y)+\overline{H(\overline{X}, Y)}+A([X, Y])T \\ &\quad -\overline{A(X)[T, \overline{Y}]}+A([X, Y])T-A(Y)A([X, T])T \\ &\quad +A(Y)[X, T]+\overline{A(Y)[\overline{X}, T]}. \quad \ldots \quad (1.26) \end{aligned}$$

Proof : From (1.16) and (1.22), we at once get (1.26).

Theorem (1.7)—We have in Grayan manifold

$$TA([\overline{X}, \overline{Y}]) = 0 \quad \ldots \quad (1.27a)$$

$$\begin{aligned} A(X)\{[\overline{T, Y}]-[T, \overline{Y}]-A([\overline{X}, Y])T &= A(Y)\{[\overline{X, T}]-[\overline{X}, T]\} \\ & A([X, \overline{Y}])T. \quad \ldots \quad (1.27b) \end{aligned}$$

Proof : Barring X and N in (1.14) and using (1.1) we obtain

$$\begin{aligned} \overline{N(\overline{X}, Y)} &= -[X, \overline{Y}]+\overline{A(X)[T, \overline{Y}]}-[\overline{\overline{X}, Y}]+[\overline{X}, \overline{Y}]-A([\overline{X}, \overline{Y}])T \\ &\quad -[X, Y]+A(X)[T, Y]+A([X, Y])T-A(X)A([T, Y])T \quad \ldots \quad (1.28) \end{aligned}$$

Using (1.14) and

$$N(T, Y) = -\overline{[T, \bar{Y}]} - [T, Y] + A([T, Y])T$$

in (1.28), we obtain

$$\overline{N(\bar{X}, Y)} = N(X, Y) - A(X)N(T, Y) - A([\bar{X}, \bar{Y}])T. \quad \ldots \quad (1.29)$$

For Grayan manifold it reduces to

$$A([\bar{X}, \bar{Y}])T = 0,$$

which is (1.27a). Barring X and Y in (1.14) and using (1.1), we obtain

$$N(\bar{X}, Y) + A(X)\{\overline{[T, Y]} - [T, \bar{Y}]\} - A([\bar{X}, Y])T$$

$$= N(X, \bar{Y}) + A(Y)\{\overline{[X, T]} - [\bar{X}, T]\} - A([X, \bar{Y}])T. \quad \ldots \quad (1.30)$$

Putting

$$N(X, Y) = 0$$

we at once obtain (1.27b).

Corollary (1.4)—We have in a Grayan manifold,

$$A([\bar{X}, \bar{Y}]) = 0 \quad \ldots \quad (1.31)$$

$$A([X, \bar{Y}]) = A(X)A([T, \bar{Y}]). \quad \ldots \quad (1.32)$$

Proof : (1.31) follows from (1.29). Barring X in (1.31) and using (1.4) and (1.1a), we obtain (1.32).

Corollary (1.5)—We also have, in a Grayan manifold

$$[\bar{X}, T] - \overline{[X, T]} = A[\bar{X}, T]T. \quad \ldots \quad (1.33)$$

$$A(X)A([T, \bar{Y}]) + A([\bar{X}, \bar{Y}]) = A(Y)A([\bar{X}, T]) - A([X, \bar{Y}]). \quad \ldots \quad (1.34)$$

Proof : Putting T for Y in (1.27b) and using (1.3), we obtain (1.33). (1.34) follows from (1.32).

Theorem (1.7)—We have in an almost Grayan manifold

$$\overline{N(X, \bar{Y})} = N(X, Y) - A(Y)N(X, T) - A([\bar{X}, \bar{Y}])T \quad \ldots \quad (1.35a)$$

$$\overline{N(X, \bar{Y})} - \overline{N(\bar{X}, Y)} = A(X)N(T, Y) - A(Y)N(X, T). \quad \ldots \quad (1.35b)$$

Proof : Barring Y and N in (1.14) and using (1.1), we obtain

$$\overline{N(X, \bar{Y})} = -\overline{[\bar{X}, Y]} + A(Y)\overline{[\bar{X}, T]} - \overline{[X, \bar{Y}]} + [\bar{X}, \bar{Y}] - A([\bar{X}, \bar{Y}])T$$

$$-[X, Y] + A([X, Y])T + A(Y)[X, T] - A(Y)A([X, T])T \quad \ldots \quad (1.36)$$

Using (1.14) and

$$N(X, T) = -[X, T]-[\bar{X}, T]+TA([X, T]). \quad \ldots \quad (1.37)$$

in (1.36), we obtain (1.35a). By subtracting (1.29) from (1.35*a*), we obtain (1.35*b*).

Corollary (1.6)—We have in an almost Grayan manifold

$$A(N(X, Y)) = A(Y)A(N(X, T))+A([\bar{X}, \bar{Y}]) \quad \ldots \quad (1.38)$$

$$A(X)A(N(T, Y)) = A(Y)A(N(X, T)). \quad \ldots \quad (1.39)$$

Proof : Using (1.4) in (1.35*a*), we obtain (1.38). Using (1.4) in (1.35*b*), we obtain (1.39).

Corollary (1.7)—We also have, in an almost Grayan manifold

$$A(N(T, Y)) = 0. \quad \ldots \quad (1.40)$$

Proof : Using (1.4) in (1.29) we obtain (1.40).

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