BLASCHKE PRODUCT FOR OPERATOR-VALUED ANALYTIC FUNCTIONS

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An operator-valued analytic function $F(z)$ in the upper half plane is said to be of bounded type if there exists a complex analytic function $\phi(z)$ such that $|\phi(z)| \leq 1$ and $\|\phi(z) F(z)\| \leq 1$ in the upper half plane. In general, the zeros of such functions are not isolated. The author studies such functions and determines their Blaschke products.

Let H be a Hilbert space and $B(H)$ be the algebra of bounded linear operators in H. A function $F(z)$ defined in some complex domain into $B(H)$ is said to be analytic if $\bar{y} F(z) x$ is complex analytic function for every pair of vectors x, y in H, where $\bar{y}$ denotes the linear functional on H corresponding to $y \in H$. The function $F(z)$ is said to have a zero at $z=w$ if $F(w)$ fails to have everywhere defined bounded inverse operator. In contrast to the complex analytic functions, the zeros of an operator-valued analytic functions are generally non-isolated. This causes a fundamental difference between the two theories, apart from the non-commutativity of the operator algebra. Hence, it is essential to develope operator-valued theory separately.

An operator-valued analytic function $F(z)$ in the upper half plane (Im $z>0$) is said to be of "bounded type" if there exists a complex analytic function $\phi(z)$ in (Im $z>0$), bounded by 1 such that $\|\phi(z) F(z)\| \leqslant 1$ in (Im $z>\phi$) ($\|.\|$ denotes the norm of the operator defined in H). In general, the existence of such majorant is not known. But if $F(z)$ takes the values in the trace class of $B(H)$ such that $(F(z) + \overline{F}(z)) \geqslant 0$ ($\bar{A}$ denotes the adjoint of A) in (Im $z>0$), then $\phi(z) = \dfrac{1}{\det(1 + F(z))}$ has the required property (de Branges 1962, p. 547). In this paper, we study the functions of bounded type and determine its Blaschke product in the upper half plane. We shall need the following extension of the Phragmén-Lindelöf principle (Hille and Phillips 1957, p. 103). Let $F(z)$ be an analytic function in the upper half plane such that $\|F(z)\|$ has continuous extension to the closed half plane and is bounded by 1 on the real axis. If

$$\liminf_{r \to \infty} \frac{1}{r} \int_0^{\pi} \log^+ \|F(re^{i\theta})\| \sin \theta \, d\theta = 0$$

where $\log^+ a = \max(\log a, 0)$, then $F(z)$ is bounded by 1 in the upper half plane.

Theorem 1—Let $\{Q_n\}$ be a sequence of projection operators and $\{z_n = x_n + iy_n\}$ be a sequence of complex numbers such that $y_n > 0$ for $n \geqslant 1$

and $\sum \frac{y_n}{x^2_n + y^2_n} < \infty$. Then the product

$$B(z) = \lim_{n \to \infty} \prod_1^n \left(1 - Q_k \frac{z}{\bar{z}_k} \right)^{-1} \left(1 - Q_k \frac{z}{z_k} \right) \tag{1}$$

converges in the operator norm uniformly on every bounded set which lies at some positive distance from $\{\bar{z}_n\}$. $B(z)$ is analytic and bounded by 1 in the upper half plane and has bounded operator inverse at every point in the upper half plane.

Proof : The proof is based on the use of the following norm inequality for operators :

$$\|AB\| \leqslant (1 + \|AB - 1\|) \leqslant (1 + \|A - 1\|)\ (1 + \|B - 1\|). \tag{2}$$

If we define,

$$\frac{1}{\rho(\bar{z})} = \sup_{k \geqslant 1} \left\| \left(\frac{1}{z} - Q_k \frac{1}{z_k} \right)^{-1} \right\|,$$

then $0 < \rho(z) < \infty$ on every bounded set which lies at some positive distance from the sequence $\{\bar{z}_n\}$. Define

$$B_n(z) = \prod_1^n \left(1 - Q_k \frac{z}{z_k} \right)^{-1} \left(1 - Q_k \frac{z}{z_k} \right).$$

Since,

$$\left\| \left(1 - Q_k \frac{z}{z_k} \right)^{-1} \left(1 - Q_k \frac{z}{z_k} \right) \right\| \leqslant 1 + \left\| \left(1 - Q_k \frac{z}{z_k} \right)^{-1} \left(1 - Q \frac{z}{z_k} \right) - 1 \right\|$$

$$\leqslant 1 + \frac{2y_k}{(x_k^2 + y_k^2)} \left\| \left(\frac{1}{z} - Q_k \frac{1}{z_k} \right)^{-1} \right\|$$

$$\leqslant \exp \frac{2}{\rho(z)} \frac{y_k}{(x_k^2 + y_k^2)}$$

hence by (2), we get,

$$\| B_n(z) \| \leqslant \exp \left(\frac{2}{\rho(z)} \sum_1^n \frac{y_k}{(x_k^2 + y_k^2)} \right).$$

Therefore, $\| B_n(z)-B_m(z) \| \leqslant \| B_m(z) \| \prod_{m+1}^{n} \left(1 + \| \left(1-Q_k \frac{z}{z_k} \right)^{-1} \times \left(1-Q_k \frac{z}{z_k} \right) - 1 \| \right)$

$$\leqslant \exp\left(\frac{2}{\rho(z)} \sum_{1}^{n} \frac{y_k}{x_k^2+y_k^2} \right) - \exp\left(\frac{2}{\rho(z)} \sum_{1}^{m} \frac{y_k}{x_k^2+y_k^2} \right).$$

As the sequence $\left\{ \frac{2}{\rho(z)} \sum_{1}^{n} \frac{y_k}{x_k^2+y_k^2} \right\}$ converges uniformly on every bounded set which lies at some positive distance from the sequence $\{\bar{z}_n\}$, it follows that $B_n(z) \to B(z)$ as $n \to \infty$ uniformly in the operator norm on every bounded set which lies at some positive distance from $\{\bar{z}_n\}$. The analyticity of $B(z)$ follows from the fact that $B(z)$ is uniform limit of analytic functions $B_n(z)$ in the upper half plane. If $z = x + iy, y > 0$ then

$$\| \left(1-Q_k \frac{z}{z_k} \right)^{-1} \left(1-Q_k \frac{z}{z_k} \right) \|^2 = \| \{(x_k - xQ_k)^2 + (Y_k + yQ_k)^2\}^{-1} \{ (x_k - xQ_k)^2 + (y_k - yQ_k)^2 \} \|$$

$$= \| 1 - \frac{4yy_k\, Q_k}{(x_k^2+y_k^2)} \{ 1 - \frac{2x\, x_k - 2y\, y_k - x^2 - y^2}{x_k^2+y_k^2} Q_k \}^{-1} \|$$

$$= \| 1 - \frac{4y\, y_k}{x_k^2+y_k^2} Q_k \{ 1 + \frac{2x\, x_k - 2y\, y_k - x^2 - y^2}{x_k^2+y_k^2} Q_K + \ldots \} \|$$

$$= \| 1 - 4y\, y_k \{(x-x_k)^2 + (y+y_k)^2\}^{-1}\ Q_k \|.$$

As $\| 1 - \alpha Q \| = \max (| 1 - \alpha | , 0)$ for all complex number α, so

$$\| B_n(z) \|^2 \leqslant \prod_{1}^{n} \| \left(1 - Q_k \frac{z}{z_k} \right)^{-1} \left(1 - Q_k \frac{z}{z_k} \right) \|^2 \leqslant 1.$$

Since $B_n(z) \to B(z)$ as $n \to \infty$ in the norm, it follows that $B(z)$ is bounded by 1 in the upper half plane. Since $\left(1 - Q_k \frac{z}{z_k} \right)^{-1} = (z - \bar{z}_k)^{-1} (z - \bar{z}_k - Q_k z)$ is well defined everywhere except at $z = \bar{z}_k$, $B_n(z)$ has bounded operator inverse at every point in the upper half plane. Therefore, for each z and sufficiently large n, $\| B_n(z)^{-1} B(z) - 1 \| < 1$, which implies that $B_n(z)^{-1} B(z)$ has bounded operator inverse in the upper half plane. The theorem follows because $B(z) = B_n(z) B_n(z)^{-1} B(z)$ has operator inverse everywhere in the upper half plane.

Example—The function $B(z)$ defined in (1) can be constructed easily with proper choices of $\{ Q_n \}$ and $\{ z_n \}$, viz. let $\{ e_n \}$ be any complete orthonormal base of the separable Hilbert space H, and Q_n be the projection whose range is the direct sum $\bigoplus_{k=1}^{n} e_k$, and $x_n = n^3$, $y_n = n$, then $\sum \frac{y_n}{x_n^2 + y_n^2} = \sum \frac{1}{1+n^2} < \infty$.

Lemma—Let $F(z)$ be an operator-valued function of bounded type in the upper half plane such that $1-F(z)$ takes completely† continuous values. If $F(w)$ has dense range in H for some w, $\operatorname{Im} w > 0$, then the zeros of $F(z)$ are atmost countable.

Proof : It follows from Theorem 18 (de Branges and Raunyak 1966, p. 353) that the zeros of $F(z)$ are isolated in the upper half plane, which implies that the zeros are atmost countable.

Theorem 2—Let $F(z)$ satisfy the hypothesis of the Lemma and suppose that if $\{ z_k = x_k + iy_k \}$ are zeros of $F(z)$, then $\sum \frac{y_k}{x_k^2 + y_k^2} < \infty$. Assume that $F(0)$ has dense range in H, then $F(z) = B(z)\, G(z)$, where $B(z)$ is the Blaschke product of the form (1) and $G(z)$ is a function of bounded type which has no zeros in $(\operatorname{Im} z > 0)$ and $1-G(z)$ takes completely continuous values.

Proof : Since $F(0)$ is the continuous extension of $F(z)$ and $1-F(z)$ is completely continuous, $1-F(0)$ is also completely continuous and so by the hypothesis, $F(0)$ has bounded operator inverse. It shows that the origin is not the limit point of zeros. If $F(z)$ has no zeros, then the theorem follows by taking $B(z)=1$ and $F(z)=G(z)$. Assume that z_1 is the point nearest to the origin such that $F(z_1)$ has no bounded operator inverse. Since $1-F(z_1)$ is completely continuous the range of $F(z_1)$ is not dense in H. Let Q_1 be the projection operator whose range is the orthogonal complement of the range of $F(z_1)$. Since $\left(1-Q_1 \frac{z}{z_1} \right)$ is analytic in the upper half plane except at $z = z_1$ and $Q_1 F(z_1)=0$,

$$F_2(z) = \left(1-Q_1 \frac{z}{z_1} \right) \left(1-Q_1 \frac{z}{z_1} \right)^{-1} F_1(z)$$

$$= \left(1-Q_1 \frac{z}{z_1} \right) F_1(z_1) + (z-z_1-Q_1 z) \frac{F_1(z)-F_1(z_1)}{z-z_1}$$

is analytic in the upper half plane, where $F(z)=F_1(z)$. Inductively, let $F_n(z)$ be the function which fails to have bounded operator inverse at z_n. Define

† An operator A is called competely continuous if the image of every bounded sequence under A contains a convergent subsequence.

$$F_{n+1}(z) = \left(1 - Q_n \frac{z}{z_n}\right) \left(1 - Q_n \frac{z}{z_n}\right)^{-1} F_n(z) \quad . (3)$$

where Q_n is the projection whose range is the orthogonal complement of the range of $F_n(z_n)$. If $F_{n+1}(z)$ has no zeros, then the theorem follows by taking $B(z) = B_n(z)$ and $G(z) = F_{n+1}(z)$. Consider the case when the process is continued and $F_n(z)$ is the sequence of functions given by (3). By the hypothesis, there exists a complex analytic function $\phi(z)$ bounded by 1 such that $\| \phi(z) F(z) \| \leqslant 1$ in the upper half plane. We show by induction that $\phi(z) F_n(z)$ is also bounded by 1. For $n = 1$, it is trivially true. Assume that $\phi(z) F_n(z)$ is bounded by 1 in the upper half plane. It follows from the proof of Theorem 1 that if $z \neq z_n$, then

$$\left\| \left(1 - Q_n \frac{z}{z_n}\right) \left(1 - Q_n \frac{z}{z_n}\right)^{-1} \right\|^2 = \left\| 1 + \frac{4yy_n}{(x-x_n)^2 + (y-y_n)^2} Q_n \right\|$$

$$= 1 + \frac{4yy_n}{(x-x_n)^2 + (y-y_n)^2} .$$

Then $\phi(z) F_{n+1}(z)$ is bounded on every bounded set which lies at some positive distance from z_n. Since $\phi(z) F_{n+1}(z)$ is continuous and therefore bounded in every bounded neighbourhood of z_n, $\phi(z) F_{n+1}(z)$ is bounded in the upper half plane,. Since on $y = h$,

$$\max_{-\infty < x < \infty} \left\| \left(1 - Q_n \frac{z}{z_n}\right) \left(1 - Q_n \frac{z}{z_n}\right)^{-1} \right\| = \frac{(h + y_n)}{|h - y_n|}$$

hence by an application of Phragmén-Lindelöf principle, it follows that $\phi(z) F_{n+1}(z)$ is bounded by $\frac{(h + y_n)}{|h - y_n|}$ in the half plane $y \geqslant h$. Since h is arbitrary, $\phi(z) F_{n+1}(z)$ is bounded by 1 in the upper half plane.

Write $F(z) = B_n(z) F_{n+1}(z)$. By Theorem 1 and the hypothesis, $B_n(z) \to B(z)$ as $n \to \infty$ in the operator norm. Thus $F(z) = B(z) G(z)$, where $G(z) = \lim_{n \to \infty} F_{n+1}(z)$ on every bounded set in the upper half plane. By construction, the zeros of $F(z)$ coincides with the zeros of $B(z)$. Hence, $G(z)$ possesses operator inverse everywhere in the upper half plane. Since by the construction and the hypothesis, Q_n has finite dimensional range for all $n \geqslant 1$, it follows by induction that $1 - F_n(z)$ takes completely continuous values in the upper half plane. So, $1 - G(z)$ is completely continuous for all z in the upper half plane. This completes the proof.

Example—Let $F_1(z)$ and $F_2(z)$ be two complex-valued analytic functions bounded by 1 in the upper half plane such that $F_1(0) \neq 0 \neq F_2(0)$. Then

$$F(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} F_1(z), & 0 \\ 0 & , \quad F_2(z) \end{bmatrix}$$

is a matrix-valued analytic function, bounded by 1 in the operator norm in the upper half plane. The zeros of $F(z)$ are precisely the zeros $\{ z_n^{(1)} = x_n^{(1)} + i y_n^{(2)} \}$ of $F_1(z)$ and the zeros $\{ z_n^{(2)} = x_n^{(2)} + i y_n^{(2)} \}$ of $F_2(z)$ Since $F_k(z)$ is bounded by 1, $k = 1,2$, $\sum \frac{y_n^{(k)}}{x_n^{(k)2} + y_n^{(k)2}} < \infty$, hence the hypothesis on the zeros in Theorem 2 holds for $F(z)$. Since $F(z)$ is matric-valued, $1 - F(z)$ takes completely continuous values. Thus $F(z)$ satisfies the hypotheses of Theorem 2.

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