EXPANSION FORMULAE FOR THE *H*-FUNCTION OF TWO VARIABLES

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The aim of the present paper is to establish five new and interesting expansion formulae for the *H*-function of two variables. Since this function is very general in nature and includes almost all the important special functions of one and two variables, a number of known and unknown expansion formulae can be obtained as special cases of our main results.

1. INTRODUCTION

The *H*-function of two variables occurring in this paper has been defined and represented (Mittal and Gupta 1972, p. 117) in the following manner :

$$H[x, y] = H\left[\begin{matrix} \begin{pmatrix} 0, n_1 \\ p_1, q_1 \end{pmatrix} \\ \begin{pmatrix} m_2, n_2 \\ p_2, q_2 \end{pmatrix} \\ \begin{pmatrix} m_3, n_3 \\ p_3, q_3 \end{pmatrix} \end{matrix} \left| \begin{matrix} (a_j; \alpha_j, A_j)_{1, p_1} \\ (b_j; \beta_j, B_j)_{1, q_1} \\ (c_j, \gamma_j)_{1, p_2} \\ (d_j, \delta_j)_{1, q_2} \\ (e_j, E_j)_{1, p_3} \\ (f_j, F_j)_{1, q_3} \end{matrix} \right| \begin{matrix} x \\ \\ y \end{matrix} \right]$$

$$= (\tfrac{1}{2}\pi i)^2 \int_{L_1} \int_{L_2} \phi(s, t)\, \theta_1(s)\, \theta_2(t)\, x^s\, y^t\, ds\, dt \qquad \ldots(1.1)$$

where

$$\phi(s, t) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j s + A_j t)}{\prod_{j=n_1+1}^{p_1} \Gamma(a_j - \alpha_j s - A_j t) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j s + B_j t)}$$

$$\theta_1(s) = \frac{\prod_{j=1}^{n_2} \Gamma(1 - c_j + \gamma_j s) \prod_{j=1}^{m_2} \Gamma(d_j - \delta_j s)}{\prod_{j=n_2+1}^{p_2} \Gamma(c_j - \gamma_j s) \prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + \delta_j s)}$$

$$\theta_2(t) = \frac{\prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j t) \prod_{j=1}^{m_3} \Gamma(f_j - F_j t)}{\prod_{j=n_3+1}^{p_3} \Gamma(e_j - E_j t) \prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + F_j t)}$$

and L_1 and L_2 are the suitable contours. Throughout the paper $(a_j; \alpha_j, A_j)_{1,\,p}$ stands for $(a_1; \alpha_1, A_1), \ldots, (a_p; \alpha_p, A_p)$ and $(a_j, \alpha_j)_{1,\,p}$ stands for $(a_1, \alpha_1), \ldots, (a_p, \alpha_p)$.

The function defined in (1.1) is an analytic function of x and y (Mittal and Gupta 1972, p. 118), if the conditions (i) and (ii) given below are satisfied :

$$\text{(i)} \quad R = \sum_1^{p_1} (\alpha_j) + \sum_1^{p_2} (\gamma_j) - \sum_1^{q_1} (\beta_j) - \sum_1^{q_2} (\delta_j) < 0$$

$$\text{(ii)} \quad S = \sum_1^{p_1} (A_j) + \sum_1^{p_3} (E_j) - \sum_1^{q_1} (B_j) - \sum_1^{q_3} (F_j) < 0$$

The integral (1.1) converges (Mittal and Gupta 1972, p. 118), if the following conditions are satisfied :

$$\text{(iii)} \quad U = \sum_1^{n_1} (\alpha_j) - \sum_{n_1+1}^{p_1} (\alpha_j) - \sum_1^{q_1} (\beta_j) + \sum_1^{m_2} (\delta_j) - \sum_{m_2+1}^{q_2} (\delta_j) + \sum_1^{n_2} (\gamma_j) - \sum_{n_2+1}^{p_2} (\gamma_j) > 0$$

$$\text{(iv)} \quad V = \sum_1^{n_1} (A_j) - \sum_{n_1+1}^{p_1} (A_j) - \sum_1^{q_1} (B_j) + \sum_1^{m_3} (F_j) - \sum_{m_3+1}^{q_3} (F_j) + \sum_1^{n_3} (E_j) - \sum_{n_3+1}^{p_3} (E_j) > 0$$

$$\text{(v)} \quad |\arg x| < (\tfrac{1}{2})\, U \pi, \quad |\arg y| < (\tfrac{1}{2})\, V \pi.$$

2. Main Results

The following new and interesting expansion formulae have been established :

First Formula

$$\sum_{k=0}^{\infty} \frac{1}{k!} H \left[\begin{array}{c|l|c} \begin{pmatrix} 0,\ n_1 \\ p_1,\ q_1+1 \end{pmatrix} & \begin{array}{l} (a_j;\ \alpha_j,\ A_j)_{1,\,p_1} \\ (b_j;\ \beta_j,\ B_j)_{1,\,q_1},\ (w-k;\ \epsilon,\ W) \end{array} & x \\ \begin{pmatrix} m_2,\ n_2+1 \\ p_2+1,\ q_2 \end{pmatrix} & \begin{array}{l} (g-k,\ \eta),\ (c_j,\ \gamma_j)_{1,\,p_2} \\ (d_j,\ \delta_j)_{1,\,q_2} \end{array} & \\ \begin{pmatrix} m_3,\ n_3+1 \\ p_3+1,\ q_3 \end{pmatrix} & \begin{array}{l} (r-k,\ \rho),\ (e_j,\ E_j)_{1,\,p_3} \\ (f_j,\ F_j)_{1,\,q_3} \end{array} & y \end{array} \right] =$$

(equation contd. p. 465)

$$
= H \left[\begin{array}{c|l|c} \binom{0,\ n_1+1}{p_1+1,\ q_1+2} & \begin{array}{l} (2+w-g-r;\ \epsilon-\eta,\ W-\rho),\ (a_j;\ \alpha_j,\ A_j)_{1,\ p_1} \\ (b_j;\ \beta_j,\ B_j)_{1,\ q_1},\ (w-g+1;\ \epsilon-\eta, W),\ (w-r+1;\ \epsilon, W-\rho) \end{array} & x \\ \binom{m_2,\ n_2+1}{p_2+1,\ q_2} & \begin{array}{l} (g,\ \eta),\ (c_j,\ \gamma_j)_{1,\ p_2} \\ (d_j,\ \delta_j)_{1,\ q_2} \end{array} & \\ \binom{m_3,\ n_3+1}{p_3+1,\ q_3} & \begin{array}{l} (r,\ \rho),\ (e_j,\ E_j)_{1,\ p_3} \\ (f_j,\ F_j)_{1,\ q_3} \end{array} & y \end{array} \right] \qquad \text{...(2.1)}
$$

where (i) $\mathrm{Re}\,(g + r - w - 1) > 0$, $\epsilon - \eta > 0$, $W - \rho > 0$ (ii) $R + \eta - \epsilon < 0$, $S + \rho - W < 0$, $U + \eta - \epsilon > 0$, $V + \rho - W > 0$ (iii) $|\arg x| < (\frac{1}{2})(U + \eta - \epsilon)\pi$, $|\arg y| < (\frac{1}{2})(V + \rho - W)\pi$.

(R, S, U, V have the same meaning as given in (1.1)).

Second Formula

$$
\sum_{k=0}^{\infty} \frac{1}{k!} H \left[\begin{array}{c|l|c} \binom{0,\ n_1}{p_1,\ q_1} & \begin{array}{l} (a_j;\ \alpha_j,\ A_j)_{1,\ p_1} \\ (b_j;\ \beta_j,\ B_j)_{1,\ q_1} \end{array} & x \\ \binom{m_2+1,\ n_2}{p_2,\ q_2+2} & \begin{array}{l} (c_j,\ \gamma_j)_{1,\ p_2} \\ (g+k,\ \mu),\ (d_j,\ \delta_j)_{1,\ q_2},\ (w-k,\ \nu) \end{array} & \\ \binom{m_3+1,\ n_3}{p_3,\ q_3+1} & \begin{array}{l} (e_j,\ E_j)_{1,\ p_3} \\ (h+k,\ \rho),\ (f_j,\ F_j)_{1,\ q_3} \end{array} & y \end{array} \right]
$$

$$
= H \left[\begin{array}{c|l|c} \binom{0,\ n_1+1}{p_1+1,\ q_1+1} & \begin{array}{l} (w+g+h;\ \mu+\nu,\ \rho),\ (a_j;\ \alpha_j,\ A_j)_{1,\ p_1} \\ (b_j;\ \beta_j,\ B_j)_{1,\ q_1},\ (w+h;\ \nu,\ \rho) \end{array} & x \\ \binom{m_2+1,\ n_2}{p_2,\ q_2+2} & \begin{array}{l} (c_j,\ \gamma_j)_{1,\ p_2} \\ (g,\ \mu),\ (d_j,\ \delta_j)_{1,\ q_2},\ (w+g,\ \mu+\nu) \end{array} & \\ \binom{m_3+1,\ n_3}{p_3,\ q_3+1} & \begin{array}{l} (e_j,\ E_j)_{1,\ p_3} \\ (h,\ \rho),\ (f_j,\ F_j)_{1,\ q_3} \end{array} & y \end{array} \right] \qquad \text{...(2.2)}
$$

where (i) $\mathrm{Re}(1 - w - g - h) > 0$

(ii) $R - \mu - \nu < 0$, $S - \rho < 0$, $U + \mu - \nu > 0$, $V + \rho > 0$

(iii) $|\arg x| < (\frac{1}{2})(U + \mu - \nu)\pi$, $|\arg y| < (\frac{1}{2})(V + \rho)\pi$.

Third Formula

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H\left[\begin{array}{c|l|c}
\begin{pmatrix} 0,\ n_1+1 \\ p_1+1, q_1+1 \end{pmatrix} & \begin{array}{l}(u-k;\rho,W),(a_j;\alpha_j,A_j)_{1,p_1}\\ (b_j;\beta_j,B_j)_{1,q_1},(w-k;\epsilon,W)\end{array} & x \\
\begin{pmatrix} m_2,\ n_2+1 \\ p_2+1,\ q_2 \end{pmatrix} & \begin{array}{l}(u-w-k,\rho-\epsilon),(c_j,\gamma_j)_{1,p_2}\\ (d_j,\delta_j)_{1,q_2}\end{array} & \\
\begin{pmatrix} m_3,\ n_3 \\ p_3,\ q_3 \end{pmatrix} & \begin{array}{l}(e_j,E_j)_{1,p_3}\\ (f_j,F_j)_{1,q_3}\end{array} & y
\end{array}\right]$$

$$= H\left[\begin{array}{c|l|c}
\begin{pmatrix} 0,\ n_1+2 \\ p_1+2, q_1+2 \end{pmatrix} & \begin{array}{l}(u;\rho,W),\left(\frac{u-1}{2};\frac{\rho}{2},\frac{W}{2}\right),(a_j;\alpha_j,A_j)_{1,p_1}\\ (b_j;\beta_j,B_j)_{1,q_1},\left(\frac{1+2w-u}{2};\frac{2\epsilon-\rho}{2},\frac{W}{2}\right),(u-1;\rho,W)\end{array} & x \\
\begin{pmatrix} m_2,\ n_2+1 \\ p_2+1,\ q_2 \end{pmatrix} & \begin{array}{l}(u-w,\rho-\epsilon),(c_j,\gamma_j)_{1,p_2}\\ (d_j,\delta_j)_{1,q_2}\end{array} & \\
\begin{pmatrix} m_3,\ n_3 \\ p_3,\ q_3 \end{pmatrix} & \begin{array}{l}(e_j,E_j)_{1,p_3}\\ (f_j,F_j)_{1,q_3}\end{array} & y
\end{array}\right] \quad \ldots(2.3)$$

where (i) $\text{Re}(u-w) > 0,\ \epsilon < \rho < 2\epsilon.$

(ii) $R + 2\rho - 2\epsilon < 0,\ S < 0,\ U + 2\rho - 2\epsilon > 0,\ V > 0.$

(iii) $|\arg x| < (\tfrac{1}{2})(U + 2\rho - 2\epsilon)\pi,\ |\arg y| < (\tfrac{1}{2}) V\pi.$

Fourth Formula

$$\sum_{k=0}^{n}\frac{(-n)_k}{(b)_k\,k!}H\left[\begin{array}{c|l|c}
\begin{pmatrix} 0,\ n_1+1 \\ p_1+1, q_1+1 \end{pmatrix} & \begin{array}{l}(u-k;\rho,W),(a_j;\alpha_j,A_j)_{1,p_1}\\ (b_j;\beta_j,B_j)_{1,q_1},(w-k;\epsilon,W)\end{array} & x \\
\begin{pmatrix} m_2+1,\ n_2 \\ p_2,\ q_2+1 \end{pmatrix} & \begin{array}{l}(c_j,\gamma_j)_{1,p_2}\\ (b+u+n-w-1+k,\rho-\epsilon),(d_j,\delta_j)_{1,q_2}\end{array} & \\
\begin{pmatrix} m_3,\ n_3 \\ p_3,\ q_3 \end{pmatrix} & \begin{array}{l}(e_j,E_j)_{1,p_3}\\ (f_j,F_j)_{1,q_3}\end{array} & y
\end{array}\right] =$$

(*equation contd. p.* 467)

$$= \frac{\Gamma(1-n-b)}{\Gamma(1-b)} H \left[\begin{array}{l|l} \binom{0,\, n_1+2}{p_1+2,\, q_1+2} & \begin{array}{l} (u;\rho, W), (b+u-1;\rho, W), (a_j;\alpha_j, A_j)_{1,p_1} \\ (b_j;\beta_j, B_j)_{1,q_1}, (w-n;\epsilon, W), (b+u+n-1;\rho, W) \end{array} \\ \binom{m_2+2,\, n_2}{p_2+1,\, q_2+2} & \begin{array}{l} (c_j, \gamma_j)_{1,p_2}, (u-w, \rho-\epsilon) \\ (b+u+n-w-1, \rho-\epsilon), (u+n-w, \rho-\epsilon), (d_j, \delta_j)_{1,q_2} \end{array} \\ \binom{m_3,\, n_3}{p_3,\, q_3} & \begin{array}{l} (e_j, E_j)_{1,p_3} \\ (f_j, F_j)_{1,q_3} \end{array} \end{array} \begin{array}{c} x \\ \\ y \end{array} \right] \quad \ldots(2.4)$$

where (i) $\rho - \epsilon > 0, R < 0, S < 0, U + 2\rho - 2\epsilon > 0, V > 0.$

(ii) $|\arg x| < (\tfrac{1}{2})(U + 2\rho - 2\epsilon)\pi, \ |\arg y| < (\tfrac{1}{2}) V\pi.$

Fifth Formula

$$\sum_{k=0}^{n} \frac{(-n)_k}{k!} H \left[\begin{array}{l|l} \binom{0,\, n_1}{p_1+1,\, q_1} & \begin{array}{l} (a_j;\alpha_j, A_j)_{1,p_1}, (v+k;\sigma, W) \\ (b_j;\beta_j, B_j)_{1,q_1} \end{array} \\ \binom{m_2,\, n_2+1}{p_2+1,\, q_2+1} & \begin{array}{l} (g-k, \eta), (c_j, \gamma_j)_{1,p_2} \\ (d_j, \delta_j)_{1,q_2}, (v+n+g-w-k-1, \eta+\sigma) \end{array} \\ \binom{m_3+1,\, n_3}{p_3,\, q_3+1} & \begin{array}{l} (e_j, E_j)_{1,p_3} \\ (w+k, W), (f_j, F_j)_{1,q_3} \end{array} \end{array} \begin{array}{c} x \\ \\ y \end{array} \right]$$

$$= H \left[\begin{array}{l|l} \binom{0,\, n_1+2}{p_1+3,\, q_1+2} & \begin{array}{l} (v+n;\sigma, W), (g+v-1;\eta+\sigma, W), (a_j;\alpha_j, A_j)_{1,p_1}, (v;\sigma, W) \\ (b_j;\beta_j, B_j)_{1,q_1}, (v;\sigma, W), (v+n+g-1;\eta+\sigma, W) \end{array} \\ \binom{m_2,\, n_2+2}{p_2+2,\, q_2+2} & \begin{array}{l} (g, \eta), (v-w, \sigma), (c_j, \gamma_j)_{1,p_2} \\ (d_j, \delta_j)_{1,q_2}, (v+g-w-1, \eta+\sigma), (v+n-w, \sigma) \end{array} \\ \binom{m_3+1,\, n_3}{p_3,\, q_3+1} & \begin{array}{l} (e_j, E_j)_{1,p_3} \\ (w, W), (f_j, F_j)_{1,q_3} \end{array} \end{array} \begin{array}{c} x \\ \\ y \end{array} \right] \quad \ldots(2.5)$$

where (i) $R < 0, S < 0, U - 2\sigma > 0, V > 0,$

(ii) $|\arg x| < (\tfrac{1}{2})(U-2\sigma)\pi, \ |\arg y| < (\tfrac{1}{2}) V\pi.$

PROOF OF (2.1) — Expressing the *H*-function of two variables occurring in the left hand side of (2.1) in terms of Mellin Barnes type contour integral by (1.1), changing the order of integration and summation which is justified due to absolute convergence of integrals involved therein, we get

$$(\tfrac{1}{2}\pi i)^2 \int_{L_1}\int_{L_2} \phi(s, t)\, \theta_1(s)\, \theta_2(t)\, x^s\, y^t \sum_{k=0}^{\infty} \frac{\Gamma(1-g+k+\eta s)\, \Gamma(1-r+k+\rho t)}{\Gamma(1-w+k+\epsilon s+Wt)\, k!}\, ds\, dt$$

$$= (\tfrac{1}{2}\pi i)^2 \int_{L_1}\int_{L_2} \phi(s,t)\, \theta_1(s)\, \theta_2(t)\, \frac{\Gamma(1-g+\eta s)\, \Gamma(1-r+\rho t)}{\Gamma(1-w+\epsilon s+Wt)}$$

$$\times\ {}_2F_1(1-g+\eta s,\ 1-r+\rho t;\ 1-w+\epsilon s+Wt;\ 1)x^s\, y^t\, ds\, dt$$

On using Gauss's theorem [Slater 1966, p. 28, (1.7.6)] for the summation of the hypergeometric series, we get the right-hand side of (2.1) with the help of the definition (1.1).

Proofs of (2.2)-(2.5) — The result (2.2) can be established similarly as above. The formula (2.3) can be proved on proceeding along the lines as given in (2.1), except here we use Kummer's theorem [Slater 1966, p. 51, (2.3.2.9)] for the summation of the hypergeometric series involved therein in place of Gauss's theorem.

Similarly to establish (2.4) and (2.5), we apply Saalschutz's theorem [Slater 1966, p. 49, (2.3.1.4)] for the summation of the corresponding hypergeometric series.

It is very interesting to note that the H-function of two variables being very general in nature and includes almost all the special functions of one and two variables such as G-function of two variables, Kampé de Fériet function, Appell functions, Fox's H-function, G-function, E-function, generalized hypergeometric function, orthogonal polynomials and other elementary functions occurring frequently in physics, applied mathematics and statistics. Consequently a large number of new and interesting results can be obtained by specializing the parameters of this function in the above results in a suitable manner.

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