FINITE ELEMENT METHOD AND VARIATIONAL GREEN FUNCTION

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(Communicated by F. C. Auluck, F.N.A.)

(*Received* 27 *March* 1973)

A finite element solution of a second order boundary value problem has been obtained by assuming the elliptic operator to be symmetric and positive definite. A theorem giving a relation between the variational approximation of the given boundary value problem and an approximate Green function is proved in this paper.

1. INTRODUCTION

The finite element method has proved to be quite successful in solving problems in the general area of structural mechanics (Gallagher 1971). Recently some attempts have been made to make use of this method in solving problems of 'Fluid Mechanics'. The central idea in the use of the finite element method is the formulation of the given problem in terms of variational integrals. Some progress has been made in discussing the convergence of the approximate solution by the finite element method and error estimates (Babuska 1970a, 1970b, 1971a, 1971b, 1972; Bramble 1970; Zlamal 1968).

In the present note, we discuss the application of the finite element method for the solution of the following boundary value problem :

$$\left.\begin{array}{ll} \text{(a)} & Lu(x) = f(x)\,, \quad x \in \Omega \\ \text{(b)} & \;\; u(x) = 0 \quad\;\;, \quad x \in \Gamma \end{array}\right\} \qquad \ldots(1)$$

where L is a second order symmetric and positive definite elliptic operator, Γ is the boundary of the closed domain Ω and $f(x)$ is a known function. Let $L_2(\Omega)$ be the space of square integrable functions on Ω such that

$$(u, v) = \int_{\Omega} u(x)\, v(x)\, dx$$

and

$$\| u \| = \{(u, v)\}^{1/2}.$$

Let $\epsilon\,(\bar{\Omega})$ be the space of all infinitely many times differentiable functions of Ω. Furthermore let $D(\Omega) \subset \epsilon\,(\Omega)$ be the space of all functions with compact support in Ω.

Let $W'(\Omega)$ be the Sobolev space of functions u together with their first partial derivatives

$$D_i u = \partial u / \partial x_i,\ (1 \leqslant i \leqslant N),\ u \text{ belonging to } L_2(\Omega).$$

$$\text{Let } (u, v)_{W'} = (u, v) + \sum_{i=1}^{n} (D_i u, D_i u)$$

and

$$\| u \|_{W'} = \{ (u, u)_{W'} \}^{1/2}$$

Let $W_0'(\Omega)$ be the closer of $D(\Omega)$ in $W'(\Omega)$ with respect to the norm $\| \cdot \|_{W'}$. The space $W_0'(\Omega)$ is the closed linear subspace of $W'(\Omega)$, the norm of u over $W_0'(\Omega)$ is given by

$$\| u \|_{W_0'} = \{ \sum_{i=1}^{n} (\| D_i u \|)^2 \}^{1/2}$$

We assume that the elliptic operator L has the following properties :

(A) The domain of L contains $D(\Omega)$ and $L(u)$ belongs to L_2 whenever u belongs to $D(\Omega)$

(B) The operator L is symmetric i.e.

$$(Lu, v) = (u, Lv),\ \forall\ u, v \in D\,(\Omega) \qquad \text{...(2)}$$

(C) The operator L satisfies the inequality (cf. Yosida 1969)

$$h_1\, (\| u \|_{W_0'})^2 \leqslant -\,(Lu, u),\ \forall\ u \in D(\Omega) \qquad \text{...(3)}$$

(D) There exists a constant h_2 such that

$$\| (Lu, v) \| \leqslant h_2 \| u \|_{W_0'} \| v \|_{W_0'},\ \forall\ u, v \in D(\Omega). \qquad \text{...(4)}$$

We assume that the function f in eqn. (1*a*) is in $L_2(\Omega)$.

2. The Finite Element Method of Solution

Given a finite dimensional subspace M of $W_0'(\Omega)$, the boundary value problem (1) can be solved by the variational approximation methods by finding a unique element $\xi^N \in M$ such that (Zlamal 1968)

$$b\,(\xi^N, \eta) = -\,(f, \eta),\ \forall\ \eta \in M. \qquad \text{...(5)}$$

This element ξ^N can be characterized in terms of minimizing functional

$$\xi^N(x) = - \int_{\Omega} G^N (x, \zeta) f(\zeta)\, d\zeta \qquad \ldots(6)$$

where $G^N (x, \zeta)$ is the finite element variational Green function. Let η_i, $1 \leqslant i \leqslant N$ be N linearly independent functions of $W_0'(\Omega)$ which form the basis of M. Consider matrix eigenvalue problem

$$Eu = \lambda Gu \qquad \ldots(7)$$

$$\text{where } E = (e_{ij}) \text{ with } e_{ij} = b(\eta_i, \eta_j),\ 1 \leqslant i,j \leqslant N \qquad \ldots(8)$$

$$G = (g_{ij}) \text{ with } g_{ij} = (\eta_i, \eta_j),\ 1 \leqslant i,j \leqslant N. \qquad \ldots(9)$$

The matrix G of (9) is symmetric and positive definite. By using (2) and (3) it can be shown that matrix E given by (8) is also symmetric and positive definite.

Finally, the approximate eigenvalues of the operator L obtained by applying the finite element method over the subspace of $W_0'(\Omega)$ are the eigenvalues λ_p^N, $1 \leqslant p \leqslant N$ of (7). Approximate eigenfunctions ξ_p^N, $1 \leqslant N \leqslant p$ are given by

$$\xi_p^N = \sum_{i=1}^{n} u_{p_i}^N \eta_i$$

where

$$u_p^N = (u_{p_1}^N \ldots\ldots u_{p_N}^N),\ 1 \leqslant p \leqslant N$$

are the eigenvectors of (7). Since the matrices E and G are positive definite and symmetric, the eigenvectors of (7) can be orthonormalized such that

$$\left.\begin{aligned} (u_p^N)^T E u_q^N &= \lambda_p^N \delta_{pq} \\ (u_p^N)^T G u_q^N &= \delta_{pq} \end{aligned}\right\} \qquad \ldots(10)$$

where $1 \leqslant p,\ q \leqslant N$.

Corresponding approximate eigenfunctions satisfy

$$\left.\begin{aligned} b(\xi_p^N, \xi_q^N) &= \lambda_p^N \delta_{pq} \\ (\xi_p^N, \xi_q^N) &= \delta_{pq} \end{aligned}\right\} \qquad \ldots(11)$$

where $1 \leqslant p, q \leqslant N$.

The relation between the eigenfunctions and the solution of the given problem (1) is given by the following theorem :

Theorem—Let ξ^N be the variational approximation to the solution of (1) over the subspace M of $W_0'(\Omega)$ with the assumptions (B), (C) and (D). Then, $\xi^N(x)$, $x \in \Omega$ is given by

$$\xi^N(x) = - \int_{\Omega} G^N(x, \zeta) f(\zeta)\, d\zeta \qquad ...(12)$$

where

$$G^N(x, \zeta) = \sum_{p=1}^{n} (\lambda_p^N)^{-1}\, \xi_p^N(x)\, \xi_p^N(\zeta), \qquad ...(13)$$

λ_p^N, $1 \leqslant p \leqslant N$, are the approximate eigenvalues of the operator L, obtained by applying finite element approximation over the same subspace M.

PROOF : Consider the equation

$$Eu = -t \qquad ...(14)$$

where E is given by (7) and

$$t = (t_1 \ldots\ldots\ldots t_N)^T$$

with $\quad t_i = (f, \eta_i),\ 1 \leqslant i \leqslant N. \qquad ...(15)$

Then, $\xi^N = \sum_{i=1}^{N} u_i^N \eta_i$ is the unique solution of (14), where

$$u^N = (u_1^N \ldots\ldots\ldots u_N^N)^T.$$

For fixed element $x \in \Omega$, let

$\eta_x = (\eta_1(x) \ldots\ldots\ldots\ldots \eta_N(x))^T$ so that

$\xi^N(x)$ given by

$$\xi^N(x) = - (\eta_x)^T E^{-1}\, t. \qquad ...(16)$$

Thus,

$$E^{-1} = \sum_{p=1}^{N} (\lambda_p^N)^{-1}\, u_p^N\, (u_p^N)^T. \qquad ...(17)$$

Hence combining (16) and (17)

$$\xi^N(x) = -\sum_{p=1}^{N} (\lambda_p^N)^{-1} (\eta_x)^T u_p^N (u_p^N)^T t.$$

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