HEAT TRANSFER OF RAREFIED GAS IN PLANE COUETTE FLOW USING ELLIPSOIDAL MODEL

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In this paper the problem of heat transfer in plane Couette flow using the ellipsoidal model is considered. A transfer equation approach is applied after assuming the approximate distribution function. Numerical results are obtained for the heat flux vector and temperature jumps, which show fairly good agreement with the results obtained by the previous authors.

INTRODUCTION

The Bhatnagar-Gross-Krooks model (1969) (or the BGK) has for several years been used by many authors as an approximation to the Boltzmann collision integral for solving problems in rarefied gas dynamics. When the Chapman-Enskog method is applied to the Boltzmann equation with this well-known model, the resulting Prandtl number is unity while the actual Prandtl number for monatomic gases is $\frac{2}{3}$. In order to combervent this shortcoming the ellipsoidal model was recently proposed by Holway (1966) and Cercignani and Tironi (1966). This new model not only retains much of the mathematical simplicity of the BGK model, but also yields the correct Prandtl number for a monatomic gas.

In the present paper Shen's (1963) method is used in solving the problem of heat transfer of a rarefied gas in plane Couette flow. In this approach the approximate form of the velocity distribution function containing some unknown coefficients with spatial and temporal variations is assumed whereas these unknown coefficients are determined by satisfying judiciously selected transfer equations. The numerical results have there been obtained for the heat flux vector and temperature jump at the two plates which have been considered at different but constant temperatures T_0 and T_s.

ANALYSIS

The linearized Boltzmann equation with the ellipsoidal model which is used in this paper is as follows (Srivastava 1970).

$$\beta\frac{\partial\phi}{\partial t} + c_i\frac{\partial\phi}{\partial x_i} = \frac{2}{(2+\lambda)\delta}\left[-\phi + n' + 2\vec{c}.\vec{q} + T'\left(c^2 - \frac{3}{2}\right) - \lambda\, c_i c_j\, p_{ij}\right] \quad ...(1)$$

where the distribution function f is assumed to be of the form

$$f(x_i, t, \vec{c}) = n_0\left(\frac{m}{2\pi k T_0}\right)^{3/2}\exp(-c^2)\,[1 + \phi\,(x_i, t, \vec{c})]. \quad ...(2)$$

Here δ is the mean free path of the gas; n' at T' the relative perturbations of the particle density n_0 and temperature T_0 respectively; $\vec{q}$ the mass velocity normalized by β^{-1}, $\beta = (2RT_0)^{-1/2}$, R the gas constant; p_{ij} the stress tensor normalized by $2n_0RT_0$; $\vec{c}$ the non-dimensional velocity vector given by $\vec{c}'$; k the Boltzmann constant; and λ a constant which can be chosen in such a way as to yield the correct Prandtl number. For a monatomic gas the relation between λ and Prandtl number is

$$P_r = \frac{2}{2+\lambda}. \quad ...(3)$$

This model will recover BGK model when λ is taken to be equal to zero ($P_r = 1$) and give $P_r = \frac{2}{3}$ for $\lambda = 1$.

In the case of steady Couette flow a monatomic gas is confined between two parallel plates and all the quantities depend only on the y-coordinate measured perpendicular to the plates. As the problem is of heat transfer the density variation is neglected ($n' = 0$). The two plates, $y = 0$ and $y = 1$, are considered at different but constant temperature T_0 and T_1 and the upper plate is moving with a relative speed Q in the x-direction. If we introduce the function $\psi(c_y, y)$ written as

$$\psi(c_y, y) = \int_{c_x=-\infty}^{\infty}\int_{c_z=-\infty}^{\infty}\left(c^2 - \frac{3}{2}\right)\exp\left[-\left(c_x^2 + c_z^2\right)\right]\phi\, dc_x dc_z$$

then equation (1) becomes

$$c_y\frac{\partial\psi}{\partial y} + \frac{2}{(2+\lambda)\delta}\psi = \frac{1}{3(2+\lambda)\pi^{1/2}\delta}\left(4c_y^4 - 4c_y^2 + 5\right)\int_{-\infty}^{\infty}\exp\left(-c_y^2\right)\psi\, dc_y \quad ...(4)$$

The boundary conditions to be satisfied by $\psi(c_y, y)$ are (Srivastava and Saraf 1971)

$$\psi^+(c_y, y = 0) = 0, \qquad c_y > 0$$

$$\psi^{-}(c_y, y = 1) = \pi \left[Q^2 + \frac{3}{2}\frac{T_1}{T_0} - \frac{3}{2}\right]\left(\frac{T_0}{T_1}\right)^{1/2} = \psi_w, \; c_y < 0 \qquad ...(5)$$

Following Shen (1963), the approximate form of the velocity distribution function which exhibits the proper collision effect is assumed as

$$\psi = \psi' \exp\left[-\frac{\alpha(y - y')}{c_y}\right] + \psi_0 \left\{1 - \exp\left[-\frac{\alpha(y - y')}{c_y}\right]\right\}$$

where y represents the distance measured along the direction of c_y, ψ' the distribution prescribed at the position ($y = 0$, $y = 1$) of the molecule having velocity c_y and the choice of the distribution ψ_0 is rather arbitrary. It should be remarked that the assumed distribution function forms the starting point of the transfer equation approach. Among the investigators employing the transfer equation approach, only Shen indicated how an approximate distribution function may be constructed taking into account the proper collision effects. For simplicity we choose the approximate distribution function to be of the form

$$\psi(c_y, y) = \left\{1 - \exp\left[-\frac{\alpha y}{c_y}\right]\right\}\sum_{n=0}^{\infty} c_y^n A_n^{+}(y) \text{ for } c_y < 0 \qquad ...(6a)$$

$$\psi(c_y, y) = \psi_w \exp\left[-\frac{\alpha(y - 1)}{c_y}\right] + \left\{1 - \exp\left[-\frac{\alpha(y - 1)}{c_y}\right]\right\}\sum_{n=0}^{\infty} c_y^n A_n^{-}(y)$$

$$\text{for } c_y > 0 \qquad ...(6b)$$

where $\alpha = \dfrac{2}{(2 + \lambda)\delta}$ and $A_n^{\pm}(y)$ are the unknown functions introduced in the approximate distribution function. Taking $n = 0$, the two unknown functions $A_0^{\pm}(y)$ are to be determined by the appropriate "transfer equations". The boundary conditions are automatically satisfied by eqn. (6).

Now we shall select two appropriate transfer equations in order to determine two unknown functions $A_0^{+}(y)$ and $A_0^{-}(y)$. These equations are formed by multiplying the collision equation (4) by the mass of the molecule and the momentum of the molecule i.e. by 1 and c_y respectively, and integrating the equation over all values of the velocity component of the molecules. These turn out to be

$$\frac{d}{dy} < c_y > = 0$$

or

$$< c_y > = - H_y, \text{ constant} \qquad ...(7)$$

$$\frac{d}{dy} < c_y^2 > = \alpha H_y$$

or
$$< c_y^2 > = - \alpha y H_y + C. \qquad ...(8)$$

To evaluate $< c_y >$ and $< c_y^2 >$ the following auxiliary functions are needed

$$Z_n(s) = \int_0^\infty \exp\left(-x^2 - \frac{s}{x}\right) x^n \, dx \qquad (n = 0, 1, 2,...). \qquad ...(9)$$

The tables for the function $Z_n(s)$ were obtained by Haung (1967) by using Gauss-Hermite quadrature. As a result eqns. (7) and (8) finally become

$$-A_0^-(y)\left[\frac{1}{2} - U_1\right] + A_0^+(y)\left[\frac{1}{2} - V_1\right] = \sqrt{\pi} F_1 \qquad ...(10)$$

$$A_0^-(y)\left[\frac{\sqrt{\pi}}{4} - U_2\right] + A_0^+(y)\left[\frac{\sqrt{\pi}}{4} - V_2\right] = \sqrt{\pi} F_2 \qquad ...(11)$$

with

$$F_1 = H_y + \frac{1}{\sqrt{\pi}} \psi_w U_1 \qquad ...(12)$$

$$F_2 = - H_y \alpha y + C - \frac{1}{\sqrt{\pi}} \psi_w V_1 \qquad ...(13)$$

and

$$U_n = Z_n\{\alpha(y - 1)\}, \; V_n = Z_n\{\alpha y\}. \qquad ...(14)$$

As the boundary conditions are automatically satisfied, $A_0^+(y)$ and $A_0^-(y)$ can only be determined as the eigenfunctions whereas the eigenvalues are the two constants H_y and C. Putting $y = 0$ in eqns. (10) and (11), we get

$$\left.\begin{aligned} -A_0^-(0)\,[v_1 - u_1] &= \sqrt{\pi} F_1(0) \\ A_0^-(0)\,[v_2 - u_2] &= \sqrt{\pi} F_2(0). \end{aligned}\right\} \qquad ...(15)$$

Eliminating $A_0^-(0)$ from eqn. (15), we have

$$-\frac{v_1 - u_1}{v_2 - u_2} = \frac{F_1(0)}{F_2(0)}. \qquad ...(16)$$

Similarly, putting $y = 1$ in eqn. (10) and (11), we have

$$\frac{v_1 - u_1}{v_2 - u_2} = \frac{F_1(1)}{F_2(1)} \qquad \ldots(17)$$

where $v_n = Z_n(0)$, $u_n = Z_n(\alpha)$. $\ldots(18)$

Using eqns. (12), (13), (16) and (17), we arrive at the following results

$$H_v = -\frac{\psi_w(v_2 - u_2)}{\sqrt{\pi[2(v_2 - u_2) + \alpha(v_1 - u_1)]}} \qquad \ldots(19)$$

$$C = -\frac{\psi_w}{\sqrt{\pi}}\left[\frac{(v_2 - u_2)\{(v_2 - u_2) + \alpha(v_1 - u_1)\}}{(v_1 - u_1)\{2(v_2 - u_2) + \alpha(v_1 - u_1)\}} - \frac{(v_2 - u_2 v_1 - u_1 v_2)}{(v_1 - u_1)}\right]. \qquad \ldots(20)$$

Solving for the unknown functions $A_0^{\pm}(y)$ from eqns. (10) and (11), we obtain

$$A_0^+(y) = \frac{\sqrt{\pi}\left[F_1\left(\frac{\pi}{4} - V_2\right) + F_2\left(\frac{1}{2} - U_1\right)\right]}{\left(\frac{1}{2} - V_1\right)\left(\frac{\pi}{4} - U_2\right) + \left(\frac{\pi}{4} - V_2\right)\left(\frac{1}{2} - U_1\right)} \qquad \ldots(21a)$$

$$A_0^-(y) = \frac{\pi\left[F_2\left(\frac{1}{2} - V_1\right) - F_1\left(\frac{\pi}{4} - V_2\right)\right]}{\left(\frac{1}{2} - V_1\right)\left(\frac{\pi}{4} - U_2\right) + \left(\frac{\pi}{4} - V_2\right)\left(\frac{1}{2} - U_1\right)} \qquad \ldots(21b)$$

The problem is thus completely solved. We shall now find out the heat flux vector and the temperature jumps.

The dimensionless heat flux vector is

$$\frac{H_v}{H_{k_n}} = \frac{(v_2 - u_2)}{(v_2 - u_2) + \frac{\alpha}{2}(v_1 - u_1)} \qquad \ldots(22)$$

where the subscript k_n denotes the quantity evaluated at the free molecular limit.

The temperature is defined as

$$T(y) = \frac{2}{3\pi^{3/2}} \int_{-\infty}^{\infty} \psi(c_y, y) \exp(-c_y^2)\, dc_y$$

$$T(y) = \frac{2}{3\pi^{3/2}}\left[\psi_w Z_0\{\alpha(y-1)\} + A_0^-(y)\left(\frac{\sqrt{\pi}}{2} - Z_0\{\alpha(y-1)\}\right) + A_0^+(y)\left(\frac{\sqrt{\pi}}{2} - Z_0\{\alpha y\}\right)\right]. \qquad \ldots(23)$$

From (23), the temperature jumps at the lower and upper plates are respectively

$$\frac{T(0) - T_0}{T_0} = \frac{2}{3\pi^{3/2}}[\psi_w u_0 + A_0^-(0)\{v_0 - u_0\}] \qquad \ldots(24)$$

$$\frac{T(1) - T_1}{T_0} = \left[1 - \frac{T_1}{T_0} + \frac{2}{3\pi^{3/2}}\{\psi_w v_0 + A_0^-(1)(v_0 - u_0)\}\right] \qquad \ldots(25)$$

where $A_0^+(1)$ and $A_0^-(0)$ can be obtained from eqn. (21) and $\alpha = \dfrac{2}{(2+\lambda)\delta}$.

The expression for $T(y)$ given in eqn. (23) is for arbitrary values of Prandtl number. We get back to the results of the BGK model by putting $\lambda = 0$ (Manocha 1972). In

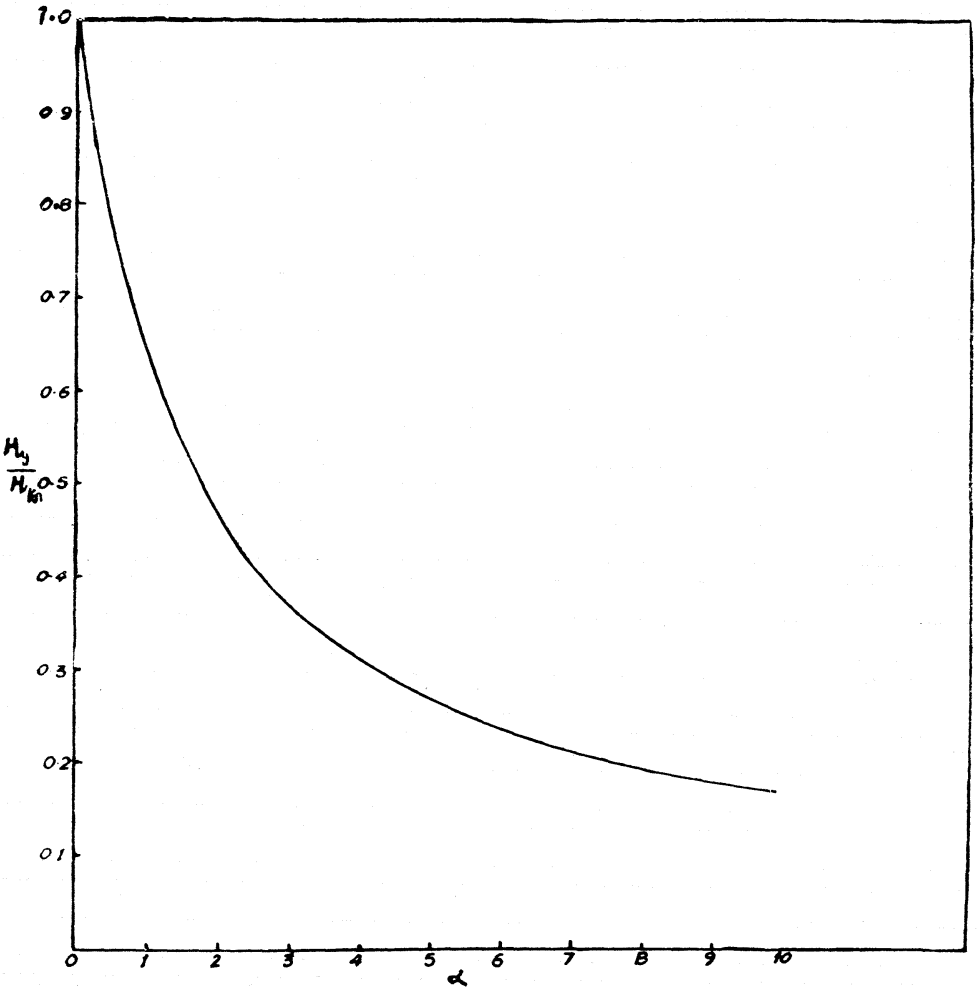

FIG. 1. Variation of heat flow with inverse Knudsen number ($\lambda = 1$).

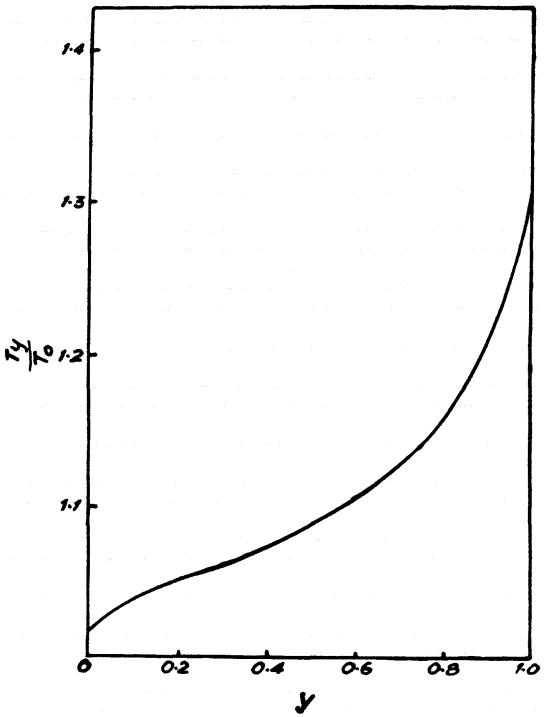

FIG. 2. Temperature profile between plates ($\lambda = 1$).

particular, we take $\delta = 10$, $Q = 0.3$ and $T_1/T_0 = 1.4$ and draw the variation of $\frac{T(y)}{T_0}$ as a function of y for $\lambda = 0$ and $\lambda = 1$ (Fig. 2). In Fig. 1, we have plotted the variation of heat flux vector with inverse of Knudsen number for $\lambda = 0$ and $\lambda = 1$. We see that our plots show fairly good agreement with the results obtained earlier (see Srivastava 1970). The results obtained from the ellipsoidal model do not deviate much from the BGK model.

Finally, we would like to mention that the transfer equation approach applied to problem of heat transfer in rarefied gas using the ellipsoidal model yields extremely satisfactory results for the reduced distribution function, the temperature profile and heat flux.

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