# APPLICATIONS OF FRACTIONAL INTEGRATION OPERATORS TO TRIPLE INTEGRAL EQUATIONS

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In the present paper formal solutions of triple integral equations involving associated Legendre functions of imaginary arguments are investigated by operational methods by making use of the technique adopted by Erdélyi (1954), Sneddon (1962) and Cooke (1965). Some new operators are introduced and applied in reducing each set of triple integral equations in terms of Fredholm integral equation of the second kind which can be solved by numerical methods. The results given recently by Srivastava (1968) are derived as particular cases.

## 1. INTRODUCTION

The following integral equations will be considered here :

$$\int_0^\infty \frac{v}{\pi} \sinh(\pi v)\, \Gamma(\tfrac{1}{2} - \beta + iv)\, \Gamma(\tfrac{1}{2} + \beta - iv)\, P^{\beta}_{-\frac{1}{2}+iv}(\cosh \alpha) f(v)\, dv$$

$$= F_1(\alpha), \quad 0 < \alpha < a \qquad \ldots(1.1)$$

$$\int_0^\infty [1 + N(v)] f(v)\, P^{\beta}_{-\frac{1}{2}+iv}(\cosh \alpha)\, dv = G_2(\alpha), \quad a < \alpha < b \qquad \ldots(1.2)$$

$$\int_0^\infty \frac{v}{\pi} \sinh(\pi v)\, \Gamma(\tfrac{1}{2} - \beta + iv)\, \Gamma(\tfrac{1}{2} + \beta - iv)\, P^{\beta}_{-\frac{1}{2}+iv}(\cosh \alpha)\, f(v)\, dv$$

$$= F_3(\alpha), \quad b < \alpha < \infty \qquad \ldots(1.3)$$

which will be referred as integral equations of the first kind, and

$$\int_0^\infty v f(v)\, P^{\beta}_{-\frac{1}{2}+iv}(\cosh \alpha)\, dv = F_1(\alpha), \; 0 < \alpha < a \qquad \ldots(1.4)$$

$$\int_0^\infty \frac{1}{\pi} \sinh(\pi v)\, \Gamma(\tfrac{1}{2} - \beta + iv)\, \Gamma(\tfrac{1}{2} + \beta - iv)\, [1 + N(v)]\, f(v) P^{\beta}_{-\frac{1}{2}+iv}(\cosh\alpha)\, dv$$

$$= G_2(\alpha),\ a < \alpha < b \qquad \text{...(1.5)}$$

$$\int_0^\infty v f(v)\, P^{\beta}_{-\frac{1}{2}+iv}(\cosh\alpha)\, dv = F_3(\alpha),\ b < \alpha < \infty \qquad \text{...(1.6)}$$

are called the triple integral equations of the second kind. Here $F$'s and $G$'s, are known functions and $f$ is an unknown to be determined. Dual and triple integral equations associated with Legendre functions are studied respectively by Bablonian (1964) and Srivastava (1968). By the applications of operators defined in this paper, the problems have been reduced to that of solving a Fredholm integral equation of the second kind which can be solved by known methods.

## 2. Results Required in the Sequel

The following integrals [Magnus *et al.* 1966, pp. 184, 425; Erdélyi *et al.* 1954, p. 330, (21)], will be found useful in the analysis that follows.

$$P^{\beta}_{-\frac{1}{2}+iv}(\cosh\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{(\sinh\alpha)^{\beta}}{\Gamma(\frac{1}{2} - \beta)} \int_0^\alpha \frac{\cos(vs)\, ds}{(\cosh\alpha - \cosh s)^{\beta+\frac{1}{2}}}$$

$$= \frac{(2\pi)^{\frac{1}{2}}}{(\sinh\alpha)^{\beta}\, \Gamma(\frac{1}{2} + \beta)\, \sinh(\pi v)\, \Gamma(\frac{1}{2} - iv + \beta)\, \Gamma(\frac{1}{2} + iv - \beta)}$$

$$\times \int_\alpha^\infty \frac{\sin(vs)\, ds}{(\cosh s - \cosh\alpha)^{\frac{1}{2}-}} \qquad \text{...(2.1)}$$

where $\mathrm{Re}(\beta) < \frac{1}{2}$, $\mathrm{Re}\,[\beta \pm (-\frac{1}{2} + iv)] > 0$.

$$\int_0^\infty P^{\beta}_{-\frac{1}{2}+iv}(\cosh\alpha) \cos(vs)\, dv = \frac{(\frac{1}{2}\pi)^{\frac{1}{2}}\, (\sinh\alpha)^{\beta}}{\Gamma(\frac{1}{2} - \beta)(\cosh\alpha - \cosh s)^{\beta+\frac{1}{2}}},\ \alpha > s$$

$$= 0,\ \alpha < s \qquad \text{...(2.2)}$$

$$\int_0^\infty P^{\beta}_{-\frac{1}{2}+iv}(\cosh\alpha) \sinh(\pi v)\, \Gamma(\tfrac{1}{2} - \beta + iv)\, \Gamma(\tfrac{1}{2} + \beta - iv) \sin(vs)\, dv$$

$$= \frac{\pi\left(\frac{\pi}{2}\right)^{\frac{1}{2}}}{(\sinh\alpha)^{\beta}\, \Gamma(\frac{1}{2} + \beta)(\cosh s - \cosh\alpha)^{\frac{1}{2}-\beta}},\ \mathrm{Re}(\beta) > -\tfrac{1}{2},\ s > \alpha$$

$$= 0,\ s < \alpha. \qquad \text{...(2.3)}$$

Generalised Mehler-Fock transform (Magnus *et al.* 1966, p. 398) is defined by

$$f(\alpha) = \int_0^\infty g(\nu)\, P^{\beta}_{-\frac{1}{2}+i\nu} (\cosh \alpha)\, d\nu \qquad \text{...(2.4)}$$

and its inversion formula is

$$g(\nu) = \frac{\nu}{\pi} \sinh (\pi\nu)\, \Gamma(\tfrac{1}{2} - \beta + i\nu)\, \Gamma(\tfrac{1}{2} + \beta - i\nu)$$

$$\times \int_0^\infty f(\alpha) \sinh \alpha\, P^{\beta}_{-\frac{1}{2}+i\nu} (\cosh \alpha)\, d\alpha. \qquad \text{...(2.5)}$$

The following variants of Abel integral equations are also needed :

$$f(x) = \int_a^x g(\alpha)(\cosh x - \cosh \alpha)^{-\delta}\, d\alpha \qquad \text{...(2.6)}$$

and

$$f(x) = \int_x^b g(\alpha)(\cosh \alpha - \cosh x)^{-\delta}\, d\alpha \qquad \text{...(2.7)}$$

have the solutions

$$g(\alpha) = \frac{\sin (\pi\delta)}{\pi} \frac{d}{d\alpha} \int_a^\alpha (\cosh \alpha - \cosh x)^{\delta-1} \sinh x\, f(x)\, dx \qquad \text{...(2.8)}$$

and

$$g(\alpha) = -\frac{\sin (\pi\delta)}{\pi} \frac{d}{d\alpha} \int_\alpha^b \sinh x\, f(x)(\cosh x - \cosh \alpha)^{\delta-1}\, dx, \qquad \text{...(2.9)}$$

$0 < \delta < 1$, respectively.

## 3. Fractional Integral Operators

In this section, we define various fractional integration operators and obtain the relations connecting them which follow directly from their definitions.

$$I^{\beta} f = \frac{1}{\Gamma(\frac{1}{2} - \beta)} \int_0^x (\cosh x - \cosh \alpha)^{-\frac{1}{2}-\beta} f(\alpha)\, d\alpha,\ R(\beta) < \tfrac{1}{2}. \qquad \text{...(3.1)}$$

$$I^{\beta^{-1}} f = \frac{1}{\Gamma(\frac{1}{2}+\beta)} \frac{d}{dx} \int_0^x (\cosh x - \cosh \alpha)^{\beta-\frac{1}{2}} \sinh \alpha \, f(\alpha) \, d\alpha,$$

$$R(\beta) > -\tfrac{1}{2}. \quad \ldots(3.2)$$

$$K^{\beta} f = \frac{1}{\Gamma(\frac{1}{2}+\beta)} \int_x^{\infty} (\cosh \alpha - \cosh x)^{\beta-\frac{1}{2}} f(\alpha) \, d\alpha, \; R(\beta) > -\tfrac{1}{2}. \quad \ldots(3.3)$$

$$K^{\beta^{-1}} f = \frac{1}{\Gamma(\frac{1}{2}-\beta)} \frac{d}{dx} \int_x^{\infty} (\cosh \alpha - \cosh x)^{-\frac{1}{2}-\beta} \sinh \alpha \, f(\alpha) \, d\alpha,$$

$$R(\beta) < \tfrac{1}{2}. \quad \ldots(3.4)$$

$$T_c^{\beta} f = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (\sinh x)^{\beta} \int_0^{\infty} f(\alpha) \cos(\alpha x) \, d\alpha \quad \ldots(3.5)$$

$$T_s^{\beta} f = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (\sinh x)^{-\beta} \int_0^{\infty} f(\alpha) \sin(\alpha x) \, d\alpha \quad \ldots(3.6)$$

$$T_p^{\beta} f = \int_0^{\infty} f(v) \, P^{\beta}_{-\frac{1}{2}+iv} (\cosh x) \, dv \quad \ldots(3.7)$$

$$T_p^{\beta^{-1}} f = \int_0^{\infty} \frac{v}{\pi} \sinh(\pi v) \, \Gamma(\tfrac{1}{2} - \beta + iv) \, \Gamma(\tfrac{1}{2} + \beta - iv) f(x) \, P^{\beta}_{-\frac{1}{2}+iv}$$

$$\times (\cosh x) \sinh x \, dx. \quad \ldots(3.8)$$

In case the limits of integration, for the operators $I^{\beta}$ and $K^{\beta}$ are, $a$ to $x$, and $a$ to $b$, respectively, then the operators will be represented by

$$\binom{x}{a} I^{\beta} f \text{ and } \binom{b}{a} K^{\beta} f \text{ respectively.}$$

The following are the interesting and useful properties of these operators which follow as a consequence of their definitions :

$$I^{\beta}\left[T_c^{\beta} f\right] = T_p^{\beta} f \quad \ldots(3.9)$$

$$K^{\beta}\left[T_s^{\beta} f\right] = T_p^{\beta}\left[\frac{1}{\pi} \Gamma(\tfrac{1}{2} - \beta + iv) \, \Gamma(\tfrac{1}{2} + \beta - iv) \sinh(\pi v) f(v)\right]$$

$$\ldots(3.10)$$

$$T_p^{\beta}\left[\frac{1}{\pi} \Gamma(\tfrac{1}{2} - \beta + iv) \, \Gamma(\tfrac{1}{2} + \beta - iv) \sinh(\pi v) \, T_s^{\beta} f\right] = K^{\beta} f \quad \ldots(3.11)$$

$$T_p^{\beta}\left[T_c^{\beta} f\right] = I^{\beta} f \qquad \text{...(3.12)}$$

$$T_s^{\beta}\left[v^{-1}\, T_p^{\beta^{-1}} f\right] = (\sinh x \sinh \alpha)^{-\beta} K^{\beta}\left[\sinh \alpha\, f(\alpha)\right] \qquad \text{...(3.13)}$$

$$T_c^{\beta}\left[\left\{\frac{v}{\pi} \sinh(\pi v)\, \Gamma(\tfrac{1}{2} + \beta - iv)\, \Gamma(\tfrac{1}{2} - \beta + iv)\right\}^{-1} T_p^{\beta^{-1}} f\right]$$
$$= (\sinh x \sinh \alpha)^{\beta} I^{\beta}\left[\sinh \alpha\, f(\alpha)\right] \qquad \text{...(3.14)}$$

$$I^{\beta^{-1}}\left[T_p^{\beta} f\right] = T_c^{\beta} f \qquad \text{...(3.15)}$$

$$K^{\beta^{-1}}\left[T_p^{\beta}\left\{\frac{1}{\pi}\, \Gamma(\tfrac{1}{2} - \beta + iv)\, \Gamma(\tfrac{1}{2} + \beta - iv) \sinh(\pi v) f(v)\right\}\right] = T_s^{\beta} f. \qquad \text{...(3.16)}$$

Two more operators $L^{\beta}$ and $M^{\beta}$ can be defined in the following manner :

$$\begin{pmatrix} x \\ d \end{pmatrix} I^{\beta^{-1}} \begin{pmatrix} f \\ e \end{pmatrix} I^{\beta} F = \begin{pmatrix} x\ f \\ d\ e \end{pmatrix} L^{\beta} F, \ (x > d \geqslant f > e) \qquad \text{...(3.17)}$$

$$\begin{pmatrix} d \\ x \end{pmatrix} K^{\beta^{-1}} \begin{pmatrix} f \\ e \end{pmatrix} K^{\beta} F = \begin{pmatrix} d\ f \\ x\ e \end{pmatrix} M^{\beta} F, \ (x < d \leqslant e < f). \qquad \text{...(3.18)}$$

By an argument similar to that used by Cooke (1965), it can be seen that

$$\begin{pmatrix} x\ f \\ d\ e \end{pmatrix} L^{\beta} F = \frac{\sin \pi(\frac{1}{2} + \beta) \sinh x}{\pi(\cosh x - \cosh d)^{\frac{1}{2}-\beta}} \int_e^f \frac{(\cosh d - \cosh s)^{\frac{1}{2}-\beta} F(s)\, ds}{(\cosh x - \cosh s)} \qquad \text{...(3.19)}$$

$$\begin{pmatrix} d\ f \\ x\ e \end{pmatrix} M^{\beta} F = \frac{\sin \pi(\frac{1}{2} + \beta) \sinh x}{\pi(\cosh d - \cosh x)^{\frac{1}{2}-\beta}} \int_e^f \frac{(\cosh s - \cosh d)^{\frac{1}{2}-\beta} F(s)\, ds}{(\cosh s - \cosh x)} \qquad \text{...(3.20)}$$

and

$$\begin{pmatrix} d \\ c \end{pmatrix} I^{\beta^{-1}} \begin{pmatrix} x \\ c \end{pmatrix} I^{\beta} F = -\begin{pmatrix} x\ d \\ d\ c \end{pmatrix} L^{\beta} F, \ (x > d > c) \qquad \text{...(3.21)}$$

$$\begin{pmatrix} e \\ d \end{pmatrix} K^{\beta^{-1}} \begin{pmatrix} c \\ x \end{pmatrix} K^{\beta} F = -\begin{pmatrix} d\ e \\ x\ d \end{pmatrix} M^{\beta} F, \ (x < d < e). \qquad \text{...(3.22)}$$

Following Sneddon (1962), we have three ranges for $\alpha$, namely $0 < \alpha < a$, $a < \alpha < b$, $b < \alpha < \infty$. We will denote these intervals by $I_1$, $I_2$ and $I_3$ respectively. Further we define any function in the interval 0 to $\infty$ as

$$f = f_1 + f_2 + f_3$$

where $f_1 = f$ in the interval $I_1$ and zero elsewhere with similar definitions for $f_2$ and $f_3$.

## 4. Solutions of Integral Equations

Using the notations of the preceding section, the integral equations of the first kind can be written in the form :

$$T_p^{\beta}\left[\frac{v}{\pi}\sinh(\pi v)\,\Gamma(\tfrac{1}{2}-\beta+iv)\,\Gamma(\tfrac{1}{2}+\beta-iv)\right]=F \qquad \text{...(4.1)}$$

$$T_p^{\beta}\Big[\{1+N(v)\}\,f(v)\Big]=G. \qquad \text{...(4.2)}$$

Assume the trial solution

$$f=T_c^{\beta}\,\phi=-\frac{1}{z}\,(\sinh z)^{2\beta}\,T_s^{\beta}\,\phi',\ \phi'=\frac{d}{ds}\,\phi(s), \qquad \text{...(4.3)}$$

provided $\lim_{s\to\infty}\phi(s)=0$

we, then, have

$$F=-(\sinh z)^{2\beta}\,T_p^{\beta}\left[\frac{1}{\pi}\sinh(\pi v)\,\Gamma(\tfrac{1}{2}-\beta+iv)\,\Gamma(\tfrac{1}{2}+\beta-iv)\,T_s^{\beta}\,\phi'\right]$$

$$=-(\sinh z)^{2\beta}\,K^{\beta}\,\phi' \qquad \text{...(4.4)}$$

$$G=T_p^{\beta}\Big[T_c^{\beta}\phi\Big]+T_p^{\beta}\Big[N(v)\,T_c^{\beta}\,\phi\Big]$$

$$=I^{\beta}\phi+T_p^{\beta}\Big[N(v)\,T_c^{\beta}\,\phi\Big]. \qquad \text{...(4.5)}$$

Hence,

$$\phi'=-(\sinh z)^{-2\beta}\,K^{\beta^{-1}}\,F,$$

we, therefore, obtain

$$\phi=(\sinh z)^{-2\beta}\,K^{\beta}\,[(\sinh\alpha)\,F]. \qquad \text{...(4.6)}$$

From (4.5), we see that

$$\phi=I^{\beta^{-1}}G-I^{\beta^{-1}}\,T_p^{\beta}\Big[N(v)\,T_c^{\beta}\,\phi\Big]$$

$$=I^{\beta^{-1}}G-T_c^{\beta}\Big[N(v)\,T_c^{\beta}\,\phi\Big]. \qquad \text{...(4.7)}$$

Similarly the triple integral equations of the second kind can be written in terms of the defined operators as

$$T_p^{\beta}\Big[vf(v)\Big]=F \qquad \text{...(4.8)}$$

$$T_p^\beta\left[\{1+N(v)\}\frac{1}{\pi}\sinh(\pi v)\,\Gamma(\tfrac{1}{2}-\beta+iv)\,\Gamma(\tfrac{1}{2}+\beta-iv)f\right]=G. \quad \text{...(4.9)}$$

If we assume the trial soultion as

$$f=T_s^\beta\,\phi=\frac{1}{v}(\sinh z)^{-2\beta}\,T_c^\beta\,\phi', \text{ provided } \phi(0)=\phi(\infty)=0, \quad \text{...(4.10)}$$

then, it gives

$$F=(\sinh z)^{-2\beta}\,T_p^\beta\left[T_c^\beta\,\phi'\right]=(\sinh z)^{-2\beta}\,I^\beta\phi' \quad \text{...(4.11)}$$

$$G=T_p^\beta\left[\frac{1}{\pi}\sinh(\pi v)\,\Gamma(\tfrac{1}{2}-\beta+iv)\,\Gamma(\tfrac{1}{2}+\beta-iv)\,T_s^\beta\,\phi\right]$$
$$+T_p^\beta\left[N(v)\frac{1}{\pi}\sinh(\pi v)\,\Gamma(\tfrac{1}{2}-\beta+iv)\,\Gamma(\tfrac{1}{2}+\beta-iv)\,T_s^\beta\,\phi\right]$$
$$=K^\beta\phi+T_p^\beta\left[N(v)\frac{1}{\pi}\sinh(\pi v)\,\Gamma(\tfrac{1}{2}-\beta+iv)\,\Gamma(\tfrac{1}{2}+\beta-iv)\,T_s^\beta\,\phi\right]. \quad \text{...(4.12)}$$

From (4.11), we get

$$\phi=(\sinh z)^{2\beta}\,I^\beta\,[(\sinh\alpha)\,F]. \quad \text{...(4.13)}$$

Also from (4.12), we have

$$\phi=K^{\beta^{-1}}G-K^{\beta^{-1}}\,T_p^\beta\left[N(v)\frac{1}{\pi}\sinh(\pi v)\,\Gamma(\tfrac{1}{2}-\beta+iv)\,\Gamma(\tfrac{1}{2}+\beta-iv)\right.$$
$$\left.\times\,T_s^\beta\,\phi\right]=K^{\beta^{-1}}G-T_s^\beta\left[N(v)\,T_s^\beta\,\phi\right]. \quad \text{...(4.14)}$$

If the equations (4.6), (4.7), (4.13) and (4.14) are evaluated on $I_1$, $I_2$ and $I_3$, we obtain six equations for the determination of six unknown functions $\phi_1$, $\phi_2$, $\phi_3$, $F_2$, $G_1$, $G_2$.

For $F_3=0$, it is possible to eliminate some of the unknown functions and reduce the number of equations to three.

To simplify (4.6) and (4.7), we suppose that

$$E=T_c^\beta\left[N(v)\,T_c^\beta\,\phi\right]$$
$$=\frac{2}{\pi}(\sinh z)^{2\beta}\int_0^\infty N(v)\cos(vx)\left[\int_0^a\phi_1(y)+\int_a^b\phi_2(y)+\int_b^\infty\phi_3(y)\right]$$
$$\times\cos(vy)\,dy\,dv$$
$$=\int_0^\infty\phi(y)\,K(x,y)\,dy \quad \text{...(4.15)}$$

where

$$K(x, y) = \frac{2}{\pi} (\sinh z)^{2\beta} \int_0^\infty N(v) \cos(vx) \cos(vy)\, dv. \qquad \ldots(4.16)$$

Hence (4.7), on $I_1$ and $I_2$, can be written as

$$\phi_1 + E = \begin{pmatrix} x \\ 0 \end{pmatrix} I^{\beta^{-1}} G_1 \qquad \ldots(4.17)$$

$$\phi_2 + E = \begin{pmatrix} a \\ 0 \end{pmatrix} I^{\beta^{-1}} G_1 + \begin{pmatrix} x \\ a \end{pmatrix} I^{\beta^{-1}} G_2. \qquad \ldots(4.18)$$

If we set $\begin{pmatrix} x \\ 0 \end{pmatrix} I^{\beta^{-1}} G_1 = \chi_1$, then it gives

$$G_1 = \begin{pmatrix} x \\ 0 \end{pmatrix} I^{\beta} \chi_1.$$

Hence the first term on the right-hand side of (4.18) is

$$-\begin{pmatrix} x & a \\ a & 0 \end{pmatrix} L^{\beta} \chi_1.$$

Evaluating (4.6) on $I_1$, $I_2$ and $I_3$, we have

$$\phi_1 = (\sinh z)^{-2\beta}\Big[\begin{pmatrix} a \\ x \end{pmatrix} K^{\beta} \{(\sinh \alpha)F_1\} + \begin{pmatrix} b \\ a \end{pmatrix} K^{\beta} \{(\sinh \alpha) F_2\} + \begin{pmatrix} \infty \\ b \end{pmatrix} K^{\beta} \{(\sinh \alpha) F_3\}\Big] \qquad \ldots(4.19)$$

$$\phi_2 = (\sinh z)^{-2\beta}\Big[\begin{pmatrix} b \\ x \end{pmatrix} K^{\beta} \{(\sinh \alpha) F_2\} + \begin{pmatrix} \infty \\ b \end{pmatrix} K^{\beta} \{(\sinh \alpha) F_3\}\Big] \qquad \ldots(4.20)$$

$$\phi_3 = (\sinh z)^{-2\beta}\Big[\begin{pmatrix} \infty \\ x \end{pmatrix} K^{\beta} \{(\sinh \alpha) F_3\}\Big]. \qquad \ldots(4.21)$$

Since $F_3$ is known, $\phi_3$ can be determined by the last equation. We now solve (4.20) for $F_2$, and substitute its value in (4.18), to get

$$(\sinh z)^{2\beta} \phi_1 = \begin{pmatrix} a \\ x \end{pmatrix} K^{\beta} F_1 - \begin{pmatrix} a & b \\ x & a \end{pmatrix} M^{\beta} \Big[(\sinh z)^{2\beta} \phi_2 - \begin{pmatrix} \infty \\ b \end{pmatrix} K^{\beta^{-1}} F_3\Big] + \begin{pmatrix} \infty \\ b \end{pmatrix} K^{\beta} F_3.$$

Rewriting these equations, we obtain

$$\phi_1 + E = \chi_1 \qquad \ldots(4.22)$$

$$\phi_2 + E = -\begin{pmatrix} x & a \\ a & 0 \end{pmatrix} L^{\beta} \chi_1 + \begin{pmatrix} x \\ a \end{pmatrix} I^{\beta^{-1}} G_2 \qquad \ldots(4.23)$$

$$(\sinh z)^{2\beta}\phi_1 = \begin{pmatrix} a \\ x \end{pmatrix} K^{\beta} F_1 - \begin{pmatrix} a & b \\ x & a \end{pmatrix} M^{\beta}\Big[(\sinh z)^{2\beta}\phi_2 - \begin{pmatrix} \infty \\ b \end{pmatrix} K^{\beta^{-1}} F_3\Big] + \begin{pmatrix} \infty \\ b \end{pmatrix} K^{\beta} F_3 \qquad \ldots(4.24)$$

$$\phi_3 = (\sinh z)^{-2\beta}\begin{pmatrix} \infty \\ x \end{pmatrix} K^{\beta} F_3. \qquad \ldots(4.25)$$

We have thus arrived at four simultaneous equations associated with four unknown functions $\phi_1$, $\phi_2$, $\phi_3$ and $\chi_1$. The case $F_1 = F_3 = 0$ gives $\phi_3 = 0$. Now we are left with only three equations which can be written in full, as

$$\phi_1(x) + \int_0^a \phi_1(y)\, K(x, y)\, dy + \int_a^b \phi_2(y)\, K(x, y)\, dy = \chi_1(x) \qquad \ldots(4.26)$$

$$\phi_2(x) + \int_0^a \phi_1(y)\, K(x, y)\, dy + \int_a^b \phi_2(y)\, K(x, y)\, dy = -\begin{pmatrix} x & a \\ a & 0 \end{pmatrix} L^{\beta} \chi_1 + \begin{pmatrix} x \\ a \end{pmatrix} I^{\beta^{-1}} G_2 \qquad \ldots(4.27)$$

$$\phi_1(x) = (\sinh z)^{-2\beta}\begin{pmatrix} a \\ x \end{pmatrix} K^{\beta} F_1 = \begin{pmatrix} a & b \\ x & a \end{pmatrix} M^{\beta}\phi_2. \qquad \ldots(4.28)$$

If the various integrations in the preceding para, in order, are denoted by the symbols $A$, $B$, $C$, $D$, $E$ and $F$, then setting $\begin{pmatrix} x \\ a \end{pmatrix} I^{\beta^{-1}} G = X$ and substituting for $\chi_1(x)$, we get the following integral equation fer $\phi_2(x)$ :

$$\phi_2 + [EF + CF + D + E\,(AF + B)]\, \phi_2 = X. \qquad \ldots(4.29)$$

This is a Fredholm integral equation of the second kind which can be best solved by numerical methods.

## 5. An Interesting Particular Case

For $\beta = 0$, the results obtained in this paper reduce to the one given earlier by Srivastava (1968).

In conclusion, it is interesting to observe that the solutions of triple integral equations associated with more general character than the associated Legendre functions can be obtained by the applications of suitable integral operators. This will form the subject matter of future communication.

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