THE RECIPROCITY FORMULA FOR DEDEKIND SUMS

S. Chowla, F.N.A. and Paul Hartung, *Institute for Advanced Study, Princeton, N. J.* 08540

(*Received* 19 *May* 1973)

An elementary proof of the reciprocity formula for Dedekind sums is deduced.

§1. Let h and k be relatively prime positive integers

$$\phi(\rho) = \rho - [\rho] - \tfrac{1}{2} \text{ if } \rho \notin Z,$$

$$\phi(\rho) = 0 \quad \text{if } \rho \notin Z.$$

The Dedekind sum is defined by

$$S(h, k) = \sum_{\rho=1}^{k} \frac{h}{k}\, \phi\left(\frac{\rho h}{k}\right).$$

The 'reciprocity' formula states that

$$\text{(A)} \qquad S(h, k) + S(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right)$$

(in particular for $h = k = 1$, both sides are 0).

§2. Consider (the variables x, y run over Z) the sum

$$S_i = \sum_{(x, y) \in D_i} X \quad (i = 0, 1, 2)$$

over all lattice points (x, y) in the 'interior' of the domains

D_0 bounded by the lines $x = 0$; $y = 0$, $x = h$, $y = k$

D_1 bounded by the lines $y = 0$, $x = k$, $y/x = h/k$

D_2 bounded by the lines $x = 0$, $y = h$, $y/x = h/k$.

A diagram shows that

$$S_0 = S_1 + S_2.$$

Now S_0 is clearly 'trivial' i.e., an easily calculated rational function of h and k.

$$S_1 = \sum_{\mu=1}^{k-1} \mu \left[\frac{\mu h}{k}\right]$$

$$= \frac{h}{k}\sum_{1}^{k-1} \mu^2 - \sum_{1}^{k-1} \mu f\left(\frac{\mu h}{k}\right)$$

where $f(x)$ denotes the fractional part of x. The 'essential' (= nontrivial) part of S_1 is, therefore,

$$-\sum_{\mu=1}^{k-1} \mu f\left(\frac{\mu h}{k}\right). \qquad \text{...(1)}$$

Now look at (a diagram helps the next step)

$$S_2 = \sum_{\nu=1}^{k-1} \{1 + 2 + 3 + \ldots + t_\nu\}$$

where $t_\nu = \left[\frac{\nu k}{h}\right]$.

So,

$$S_2 = \frac{1}{2}\sum_{\nu=1}^{h-1} \left(\left[\frac{\nu k}{h}\right]^2 + \left[\frac{\nu k}{h}\right]\right)$$

$$= p + q.$$

Writing $[x] = x - f(x)$ we see that q is 'trivial' $\left(\text{use} \sum_{1}^{h-1} f\left(\frac{rk}{h}\right) = \sum_{1}^{h-1} \frac{\nu}{h} = \frac{h-1}{2}\right)$

while the non-trivial part of p is

$$-\sum_{\nu=1}^{h-1} \frac{\nu k}{h} f\left(\frac{\nu k}{h}\right). \qquad \text{...(2)}$$

Collecting together the nontrivial parts of S_0, S_1, S_2 which are respectively 0, (1) and (2), we obtain the result that

$$\sum_{\mu=1}^{k-1} \frac{\mu}{k} f\left(\frac{\mu h}{k}\right) + \sum_{\nu=1}^{h-1} \frac{\nu}{k} f\left(\frac{\nu k}{h}\right)$$

is 'trivial', whence the statement (A) of the 'reciprocity law.'

§3. It is clear now that a '1-line' proof of the reciprocity law is

Write $S_i = \sum_{(x, y) \varepsilon D_i} x$. Use $S_0 = S_1 + S_2$.

This is similar to Eisenstein's geometrical proof of one of ten proofs of Gauss of the law of quadratic reciprocity.

DYNAMIC RESPONSE OF CIRCULAR PLATES ON ELASTIC FOUNDATION SUBJECTED TO SONIC BOOMS

by RANESH SIRCAR, *Department of Mathematics, Regional Engineering College, Durgapur* 9

(Communicated by F. C. Auluck, F.N.A.)

(*Received* 2 *November* 1972; *after revision* 2 *April* 1973)

The dynamic response of circular plates resting on elastic foundation subjected to an N-shaped pressure pulse resulting from a sonic boom is presented here. Case of a plate supported on Winkler type foundation is considered and solution obtained. The solution is given for a plate restrained at the outer boundary by means of supports which resist elastically both the transverse displacements and rotations. The plate is analysed according to classical plate theory, neglecting rotatory inertia and shear strains.

1. INTRODUCTION

The dynamic theory of plates finds many applications in modern technology. Free and forced vibrations of circular elastic plates have been studied by various investigators. In more recent times Reismann (1959) treated the case of harmonically oscillating concentrated loads applied to a clamped circular plate while Weiner (1965 a, b) considered the forced axisymmetric vibration and transient motion of circular plates subjected to impulsive and moving loads, the outer boundary of the plate being supposed to be elastically built-in. Parnes (1970) studied the axisymmetric dynamic response of circular plates subjected to sonic booms; the solution was obtained for a plate with axisymmetric but arbitrary linear restraints at the outer boundary (the edge restraints are with respect to both rotational and transverse motions).

In this investigation the dynamic response of circular plates resting on elastic foundation (Winkler type) subjected to an N-shaped pressure wave resulting from a sonic boom, is presented. The solution is obtained for a plate with arbitrary (linear) elastic restraints at the edge against both rotational and transverse displacements. Typical centre displacement-time variation graph is given for a circular plate having elastic rotational restraints at the edge but with rigid transverse supports.

1. MATHEMATICAL FORMULATION OF THE PROBLEM AND ITS BASIC SOLUTION

Forced axisymmetric transverse motion of a homogeneous, isotropic elastic plate of constant thickness subjected to time-dependent axisymmetric load $P(r, t)$ are governed by the partial differential equation

$$D \nabla^4 w(r, t) + \rho h \ddot{w}(r, t) = P(r, t) \qquad \ldots(1.1)$$

where $w(r, t)$ is the deflection function,

$$D = \frac{Eh^3}{12(1-\nu^2)}, \text{ flexural rigidity of the plate}$$

E = modulus of elasticity

h = thickness of the plate

ρ = mass density of the plate

and dots denote differentiation with respect to time t; under the restriction that the deflections are small in comparison with the plate thickness and that the shear deflections, rotatory inertia and damping can be neglected.

If the plate is supposed to be resting on Winkler type elastic foundation, the equation of motion (1.1) takes the following form (Timoshenko and Krieger 1959)

$$D \nabla^4 w(r, t) + kw(r, t) + \rho h \ddot{w}(r, t) = P(r, t) \quad \text{...(1.2)}$$

where k is the foundation stiffness constant.

Introducing a non-dimensional variable

$$\eta = \frac{r}{a}$$

a being the radius of the plate ; the equation (1.2) is written as

$$\nabla^4 w(\eta, t) + \frac{ka^4}{D} w(\eta, t) + \frac{\rho h}{D} a^4 \ddot{w}(\eta, t) = \frac{a^4}{D} P(\eta, t) \quad \text{...(1.3)}$$

where $\nabla^4 = \nabla^2 . \nabla^2$,

$$\nabla^2 \equiv \frac{\partial^2}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta}$$

Boundary and Initial Conditions

The plate is restrained at the outer boundary $r = a$ by means of supports which resist elastically both transverse displacements and rotations. The boundary conditions of the plate are thus

$$\left.\begin{aligned}
&\text{(i)} \quad w(0, t) \text{ must be finite,}\\
&\text{(ii)} \quad w'(0, t) = 0\\
&\text{(iii)} \quad \frac{D}{\alpha a^3}\left\{ w''' + \frac{w''}{\eta} - \frac{w'}{\eta^2}\right\}_{\eta=1} = w(1, t)\\
&\text{(iv)} \quad -\frac{D}{\beta a}\left\{ w'' + \frac{\nu}{\eta} w'\right\}_{\eta=1} = w'(1, t)
\end{aligned}\right\} \quad \text{...(1.4)}$$

where primes indicate differentiation with respect to η. α and β are the linear elastic restraints at the edge of the plate against transverse and rotational displacements respectively.

The initial conditions are prescribed as

$$w(\eta, 0) = \dot{w}(\eta, 0) = 0. \qquad ...(1.5)$$

We first consider the free vibrations of the plate. Assuming free vibrations of the form

$$w(\eta, t) = f(\eta)\, e^{i\bar{\omega}t} \qquad ...(1.6)$$

the resulting homogeneous equation of motion is

$$\left\{\left(\frac{d^2}{d\eta^2} + \frac{1}{\eta}\frac{d}{d\eta}\right)^2 - \left(\frac{\rho h a^4}{D}\bar{\omega}^2 - \frac{k a^4}{D}\right)\right\} f(\eta) = 0. \qquad ...(1.7)$$

The general solution of (1.7) is

$$f(\eta) = AJ_0(\lambda\eta) + BI_0(\lambda\eta) + CY_0(\lambda\eta) + DK_0(\lambda\eta) \qquad ...(1.8)$$

in which

$$\lambda^2 = \left\{\frac{\rho h\omega^2 - k}{D}\right\}^{\frac{1}{2}} \times a^2 \qquad ...(1.9)$$

J_n, Y_n are Bessel functions of first and second kind, respectively; and I_n, K_n the modified Bessel function of first and second kind, respectively.

Making use of the conditions of the bounded displacement and axial symmetry at $\eta = 0$, we have

$$f(\eta) = AJ_0(\lambda\eta) + BI_0(\lambda\eta) \qquad ...(1.10)$$

Using the last two boundary conditions of (1.4), we have

$$\left.\begin{aligned} &A[J_0(\lambda) - \Gamma_s\lambda^3 J_1(\lambda)] + B[I_0(\lambda) - \Gamma_s\lambda^3 I_1(\lambda)] = 0 \\ &A[\lambda J_0(\lambda) + (\Gamma_R - 1)\, J_1(\lambda)] + B[-\lambda I_0(\lambda) - (\Gamma_R - 1)\, I_1(\lambda)] = 0 \end{aligned}\right\} \qquad ...(1.11)$$

in which

$$\Gamma_s \equiv \frac{D}{\alpha a^3}, \quad \Gamma_R \equiv \left(\frac{\beta a}{D} + \nu\right)$$

are non-dimensional constants.

For a non-trivial solution to exist, the determinant of the coefficient must vanish, from which the frequency equation is given as

$$\begin{aligned} &2\,\lambda J_0(\lambda)\, I_0(\lambda) - 2\,\Gamma_s(\Gamma_R - 1)\,\lambda^3 J_1(\lambda)\, I_1(\lambda) \\ &\quad + (\Gamma_R - 1 - \Gamma_s\lambda^4)\,[J_0(\lambda)\, I_1(\lambda) + J_1(\lambda)\, I_0(\lambda)] = 0 \end{aligned} \qquad ...(1.12)$$

The roots λ_i $(i = 1, 2, \dots)$ are the eigenvalues, while the corresponding eigenfunctions representing the mode shapes become

$$f_i(\eta) = \{I_0(\lambda_i) - \Gamma_s \lambda_i^3 I_1(\lambda_i)\} J_0(\lambda_i \eta) - \{J_0(\lambda_i) - \Gamma_s \lambda_i^3 J_1(\lambda_i)\} I_0(\lambda_i \eta). \quad \dots(1.13)$$

2. Solution of the Problem for Axisymmetric Time-varying Pressure Pulse

The forced displacements $w(\eta, t)$ produced by axisymmetric forces, are now assumed to be represented by the infinite series

$$w(\eta, t) = \sum_{j=1}^{\infty} f_j(\eta)\, g_j(t) \quad \dots(2.1)$$

where

$$f_j(\eta) = \{I_0(\lambda_j) - \Gamma_s \lambda_j^3 I_1(\lambda_j)\} J_0(\lambda_j \eta) - \{J_0(\lambda_j) - \Gamma_s \lambda_j^3 J_1(\lambda_j)\} I_0(\lambda_j \eta). \quad \dots(2.2)$$

Substituting (2.1) in (1.3) and interchanging the order of differentiation and summation gives

$$\sum_j f_j(\eta) \left\{ g_j(t) \left(\frac{d^2}{d\eta^2} + \frac{1}{\eta}\frac{d}{d\eta} \right)^2 + \frac{ka^4}{D} g_j(t) + \frac{\rho h a^4}{D} \ddot{g}_j(t) \right\}$$

$$= \frac{a^4}{D} P(\eta, t). \quad \dots(2.3)$$

Exploiting the fact that $f_j(\eta)$ satisfies the following homogeneous ordinary differential equation

$$\left[\left(\frac{d^2}{d\eta^2} + \frac{1}{\eta}\frac{d}{d\eta} \right)^2 - \lambda_j^4 \right] f_j(\eta) = 0 \quad \dots(2.4)$$

where

$$\lambda_j^2 = \left\{ \frac{\rho h \omega_j^2 - k}{D} \right\}^{\frac{1}{2}} a^2$$

eqn. (2.3) takes the form

$$\sum_j \left\{ g_j(t)\, \lambda_j^4 + \frac{ka^4}{D} g_j(t) + \frac{\rho h a^4}{D} \ddot{g}_j(t) \right\} f_j(\eta)$$

$$= \frac{a^4}{D} P(\eta, t). \quad \dots(2.5)$$

Multiplying both sides of (2.5) by $\eta f_i(\eta)\, d\eta$ and integrating with respect to η fram 0 to 1, we have

$$\left\{ \ddot{g}_j(t) + \frac{D}{\rho h a^4}\left(\lambda_j^4 + \frac{ka^4}{D}\right) g_j(t) \right\} \mu_j$$

$$= \frac{1}{\rho h a}\int_0^1 \eta\, P(\eta, t) f_i(\eta)\, d\eta \qquad \ldots(2.6)$$

where

$$\mu_j = \Big\{ \tfrac{1}{2}\, [J_0^2(\lambda_j) + J_1^2(\lambda_j)]\, [I_0(\lambda_j) - \Gamma_s \lambda_j^3\, I_1(\lambda_j)]^2$$

$$+ [J_0(\lambda_j) - \Gamma_s \lambda_j^3\, J_1(\lambda_j)]^2\, [I_0^2(\lambda_j) - I_1^2(\lambda_j)] + \frac{\Gamma_s \lambda_j^2}{2}$$

$$\times [J_1(\lambda_j)\, I_0(\lambda_j) - I_1(\lambda_j)\, J_0(\lambda_j)]^2$$

$$+ \frac{2\,(\Gamma_R - 1)}{(\Gamma_R - 1 - \Gamma_s \lambda_j^4)^2}\, [J_0(\lambda_j)\, I_0(\lambda_j) - \Gamma_s^2\, \lambda_j^6\, J_1(\lambda_j)\, I_1(\lambda_j)]^2 \Big\}$$

[vide Parnes 1970]. ...(2.7)

The uncoupled equation on $g_j(t)$ becomes

$$\ddot{g}_j(t) + \bar{\omega}_j^2\, g_j(t) = \frac{G_j(t)}{\rho h \mu_j} \qquad \ldots(2.8)$$

where

$$G_j(t) = \int_0^1 f_j(\eta)\, P(\eta, t)\, \eta\, d\eta. \qquad \ldots(2.9)$$

The general solution of (2.8) for the prescribed initial conditions (1.5), is

$$g_j(t) = \frac{1}{\rho h \mu_j \bar{\omega}_j}\int_0^t G_j(t)\, \sin \bar{\omega}_j(t - \tau)\, d\tau \qquad \ldots(2.10)$$

and the deflections are given by

$$w(\eta, t) = \sum_{j=1}^{\infty} f_j(\eta)\, g_j(t). \qquad \ldots(2.11)$$

It may be noted that the solution degenerates to that given by Weiner (1965) by letting $\Gamma_s = 0$ for a plate fully constrained against transverse motion at the edge.

3. Solution for Uniformly Distributed Time-dependent, Axisymmetric Loads : Response to Sonic Boom

For the case of applied loads $P(t)$ which are not space-dependent, the expression for $G_j(t)$ becomes, upon substitution of (1.13) into (2.9),

$$G_j(t) = P(t)\, Q_j(\lambda_j) \qquad \text{...(3.1)}$$

where

$$Q_j(\lambda_j) = \frac{1}{\lambda_j}\Big\{[I_0(\lambda_j) - \Gamma_s\lambda_j^3\, J_1(\lambda_j)]\, J_1(\lambda_j) - [J_0(\lambda_j) - \Gamma_s\lambda_j^3\, I_1(\lambda_j)]\, I_1(\lambda_j)\Big\}. \qquad \text{...(3.2)}$$

We would consider the response of the plate to a particular pressure pulse $P(t)$, resulting from a sonic boom. It has been shown that the applied pressure-time history at far fields due to an aircraft flying at a supersonic velocity V (for which the effect of the boom is felt over a distance L_s) may be approximated closely by means of an N-shaped pulse (Parnes 1970)

$$P(t) = P_0\left(1 - \frac{2t}{T^*}\right),\ t < T^* \qquad \text{...(3.3)}$$

where T^*, the time-duration of the pressure, is given by

$$T^* = \frac{L_s}{V}. \qquad \text{...(3.4)}$$

Introducing the non-dimensional variables

$$\left.\begin{aligned} \zeta &= \frac{t}{T^*},\quad \bar{g}_j = \frac{E}{P_0 h}\, g_j \\ \xi_j &= \bar{\omega}_j\, T,\ \bar{k} = \frac{ka}{E}. \end{aligned}\right\} \qquad \text{...(3.5)}$$

Substituting (3.1) and (3.3) in (2.10) and performing the indicated integration, results in the following expression for $\bar{g}_j$

$$\bar{g}_j = \frac{12\,(1-\nu^2)\, Q_j\,(\lambda_j)\left(\dfrac{a}{h}\right)^4 F\,(\zeta)}{\mu_j\left\{\lambda_j^4 + 12\,(1-\nu^2)\,\bar{k}\left(\dfrac{a}{h}\right)^3\right\}} \qquad \text{...(3.6)}$$

where
$$\begin{aligned} F\,(\zeta) &= 1 - \cos\xi_j\,\zeta - \frac{2}{\xi_j}\,(\xi_j\,\zeta - \sin\xi_j\,\zeta),\ \zeta \leqslant 1 \\ &= \cos\xi_j\,(\zeta - 1) - \cos\xi_j\,\zeta - \frac{2}{\xi_j}\,[\sin\xi_j(\zeta - 1) - \sin\xi_j\,\zeta],\ \zeta \geqslant 1 \end{aligned} \qquad \text{...(3.7)}$$

and $$\xi_j = \left[\lambda_j^4 + 12(1-\nu^2)\bar{k}\left(\frac{a}{h}\right)^3\right]^{\frac{1}{2}}\left(\frac{h}{a}\right) T^*\left[\frac{E}{12(1-\nu^2)\rho}\right]^{\frac{1}{2}} \quad ...(3.8)$$

Noting that the expression for the velocity of longitudinal waves in a plate

$$C_L = \left\{\frac{E}{12(1-\nu^2)\rho}\right\}^{\frac{1}{2}} \quad ...(3.9a)$$

appear in the expression (3.8) and defining the critical time required for a wave to travel the distance of one radius as

$$T_{cr} = \frac{a}{C_L} \quad ...(3.9b)$$

it is observed that the functions $\bar{g}_j$ depend solely on a set of non-dimensional variables

$$\bar{g}_j = \bar{g}_j\left[\frac{h}{a}, \nu, \bar{k}, \Gamma_s, \Gamma_R, \frac{T^*}{T_{cr}}, \zeta\right]. \quad ...(3.10)$$

The resulting non-dimensional deflections are then

$$\bar{w}(\eta, \zeta) = \frac{Ew}{P_0 h} = \sum_{j=1}^{\infty} f_j(\eta)\, \bar{g}_j(\zeta). \quad ...(3.11)$$

4. Numerical Result

Typical centre displacement-time variation graph for an N-pulse in put has been drawn for a circular plate with rigid transverse support ($\Gamma_s = 0$) and having no rotational restraint ($\Gamma_R = \nu$) at the edge. The result is given for a plate with

$$\nu = 0.25, \ \frac{h}{a} = 8.33 \times 10^{-3}, \ \bar{k} = 12 \times 10^{-10}, \ \frac{T^*}{T_{cr}} = 700.$$

Table of eigenvalues (Table I) were obtained in course of the search for the roots of eqn. (1.2), employing a digital computer. The centre displacement-time variation graph was traced with the help of calculations involving the use of only first three or four terms of the series solution. Contribution made by the second and third term of the series is small compared to the first term.

5. Conclusions

The largest dynamic deflection due to the sonic boom occurs at the centre of the plate. The centre displacement-time graph (Fig. 1) of the plate indicates that the centre of the plate does not only produce the largest dynamic deflection but also tends to build up with time and has a relative maximum amplitude in each half of the sonic boom.

TABLE I

Table of eigenvalues

λ_j \ Γ_R	0	0.25	1.0	10	100
λ_1	2.1080	2.2046	2.4048	2.9529	3.1652
λ_2	5.4188	5.4463	5.5201	5.9280	6.2471
λ_3	8.5920	8.6082	8.6573	8.9762	9.3534
λ_4	11.747	11.7586	11.7915	12.0570	12.4657
λ_5	14.896	14.9051	14.9309	15.1560	15.5813
λ_6	18.043	18.0499	18.0711	18.2662	18.6990
λ_7	21.187	21.1936	21.2116	21.3838	21.8184
λ_8	24.331	24.3368	24.3525	24.5064	24.9393
λ_9	27.475	27.4796	27.4936	27.6325	28.0616
λ_{10}	30.618	30.6167	30.6243	30.7576	31.1808

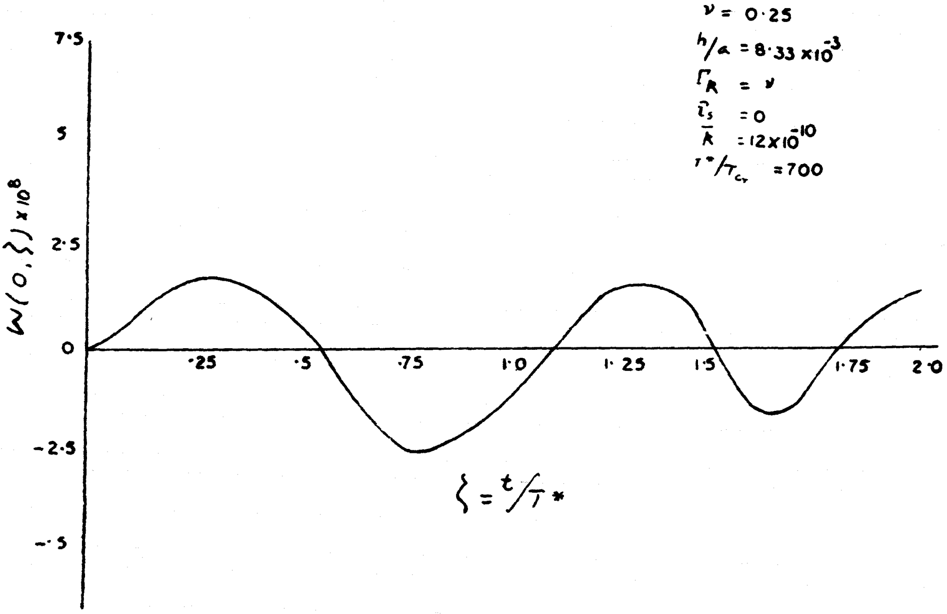

FIG. 1. Centre displacement-time variation.

The factors affecting the dynamic deflection are : (i) the increase in the period of the sonic boom and (ii) the increase in the overpressure of the boom. It has been observed that the increase in the period will result in increased dynamic deflection of the plate, and this can well form a part of a detailed investigations in the field of structural dynamics. The second factor viz., the effect of overpressure can be equated with that of static loading of the plate. The increase in the overpressure of the sonic boom will increase the dynamic amplitude of the plate but without any dynamic amplification (additional) if the boom period remains the same. However, increase in boom period will produce large structural response to the boom and increase in boom overpressure will intensify the community noise levels.

References

Parnes, R. (1970). Dynamic response of circular plates subjected to sonic booms. *Int. J. Solids Struct.*, **6**, 21-36.

Reisman, H. (1959). Forced vibration of a circular plate. *J. appl. Mech.*, **26**, 526-27.

Timoshenko, S., and Krieger, S. W. (1959). Theory of Plates and Shells. McGraw-Hill Book Co., Inc., New York.

Weiner, R. S. (1965a). Forced axisymmetric motions of circular elastic plates. *J. appl. Mech.*, **32**, 893-98.

——— (1965b). Transient Motion of circular elastic plates subjected to impulsive and moving loads. *Bell System Tech. J.*, **44**, 2409-37