A SIMILARITY SOLUTION FOR THE FLOW OF TWO IMMISCIBLE FLUIDS WITH MEAN PRESSURE

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The present paper analytically discusses the phenomena of flow of two immiscible liquids through homogeneous porous media with capillary pressure by employing a similarity technique. This problem has great importance in petroleum technology. The underlying basic assumption made in the present analysis is that the individual pressure of the two flowing phases may be replaced by their common mean pressure. The mathematical formulation leads to a nonlinear differential equation which has been reduced to the well-known Abel's equation of the second kind by using a similarity technique. A formal mathematical solution of the later has been obtained in terms of transcendental functions.

1. Introduction

The oil-water movement in a porous medium is an important problem of petroleum technology and water hydrology (Scheidegger 1966). The motion of two immiscible liquids in a homogeneous porous medium was obtained by Buckley and Leverett (1942) without considering capillary pressure. The other authors have discussed this problem from different view points; for example, Hovnassian (1961), McEwen (1959), Fayers (1959), Douglas *et al.* (1959) and Verma (1966, 1968a, b, 1969a, b). Most of the above authors, except Verma have given some type of numerical solutions to this problem with capillary pressure effect and only recently Verma has solved these problems by the perturbation method.

In this paper we assume that the individual pressure of the two flowing phases may be replaced by their common mean pressure (Oroveanu 1964) and the expression for the phase saturation distribution are obtained. The mathematical formulation leads to a non-linear differential equation which has been reduced to the well-known Abel's equation of the second kind (Murphy 1969) by using a similarity technique. A formal mathematical solution of the problem has been obtained in terms of transcendental functions (Hansen 1964).

We consider here that the injection of water into an oil formation in porous medium furnishing a two phase liquid-liquid flow problem.. Such a problem is generally encountered in Petroleum technology, for example, secondary recovery process, and in ground-water hydrology, for example, replenishment problems.

Our particular interest in the present paper is to obtain an analytical expression for phase saturation by using a similarity technique for the case of isotropic porous media. The mathematical solution of the nonlinear differential equation of the two immiscible liquids flow has been formally obtained.

2. Mathematical Formulation of the Problem

The seepage velocity of water (V_w) and oil (V_o) may be written as (by Darcy's law)

$$V_w = -\frac{K_w}{r_w} K \frac{\partial p_w}{\partial x}, \qquad \text{...(2.1)}$$

$$V_o = -\frac{K_o}{r_o} K \frac{\partial p_o}{\partial x}, \qquad \text{...(2.2)}$$

where K is the permeability of the homogeneous medium, K_w and K_o are the relative permeability of water and oil, which are functions of S_w and S_o (S_w and S_o are the saturations of water and oil) respectively, p_w and p_o denote the pressure of water and oil, while r_w and r_o are the constant kinematic viscosities of the phases.

The equations of continuity (phase densities are constant) are

$$m \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \qquad \text{...(2.3)}$$

$$m \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \qquad \text{...(2.4)}$$

where m is the porosity of medium. From the definition of phase saturation, it is evident that

$$S_w + S_o = 1. \qquad \text{...(2.5)}$$

The capillary pressure (p_c), defined as the pressure discontinuity of the flowing phases across their common interface may be written as

$$p_c = p_o - p_w, \qquad \text{...(2.6)}$$

The equation of motion for saturation can be obtained by substituting the values of V_w and V_o from eqns. (2.1) and (2.2) in eqns. (2.3) and (2.4) respectively, we get,

$$m \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{K_w}{r_w} K \frac{\partial p_w}{\partial x}\right), \qquad \text{...(2.7)}$$

$$m \frac{\partial S_o}{\partial t} = \frac{\partial}{\partial x}\left(\frac{K_o}{r_o} K \frac{\partial p_o}{\partial x}\right). \qquad \text{...(2.8)}$$

Eliminating $\frac{\partial p_w}{\partial x}$ from eqns. (2.6) and (2.7), we obtain

$$m\frac{\partial S_w}{\partial t}=\frac{\partial}{\partial x}\left[\frac{K_w}{r_w}K\left(\frac{\partial p_o}{\partial x}-\frac{\partial p_c}{\partial x}\right)\right]. \qquad ...(2.9)$$

Combining eqns. (2.8) and (2.9) and using eqn. (2.5), we have

$$\frac{\partial}{\partial x}\left[\left(\frac{K_w}{r_w}K+\frac{K_o}{r_o}K\right)\frac{\partial p_o}{\partial x}-\frac{K_w}{r_w}K\frac{\partial p_c}{\partial x}\right]=0. \qquad ...(2.10)$$

Integrating eqn. (2.10) with respect to x, we get

$$\left(\frac{K_w}{r_w}K+\frac{K_o}{r_o}K\right)\frac{\partial p_o}{\partial x}-\frac{K_w}{r_w}K\frac{\partial p_c}{\partial x}=-V \qquad ...(2.11)$$

where V is a constant of integration which can be evaluated later on.

By simplifying (2.11), we get

$$\frac{\partial p_o}{\partial x}=\frac{1}{1+\frac{K_o}{K_w}\cdot\frac{r_w}{r_o}}\frac{\partial p_c}{\partial x}-\frac{V}{\left(\frac{K_w}{r_w}K+\frac{K_o}{r_o}K\right)}. \qquad ...(2.12)$$

Putting the value of $\frac{\partial p_o}{\partial x}$ in eqn. (2.9), we have

$$m\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial x}\left[\frac{K_o/r_o}{1+\frac{K_o}{K_w}\cdot\frac{r_w}{r_o}}\frac{\partial p_c}{\partial x}+\frac{V}{1+\frac{K_o}{K_w}\cdot\frac{r_w}{r_o}}\right]=0. \qquad ...(2.13)$$

The value of the pressure of oil (p_o) can be written as (Oroveanu 1966)

$$p_o=\frac{p_o+p_w}{2}+\frac{p_o-p_w}{2}=\bar{p}+\tfrac{1}{2}p_c \qquad ...(2.14)$$

where $\bar{p}$ is the mean pressure which is constant and

$$\frac{\partial p_o}{\partial x}=\frac{1}{2}\frac{\partial p_c}{\partial x}. \qquad ...(2.15)$$

Substituting this value of $\frac{\partial p_o}{\partial x}$ in eqn. (2.11), we get

$$V=\tfrac{1}{2}\left[\left(\frac{K_w}{r_w}K-\frac{K_o}{r_o}K\right)\right]\frac{\partial p_c}{\partial x}. \qquad ...(2.16)$$

Then by substituting the value of V from eqn. (2.16) into (2.13) we obtain

$$m\frac{\partial S_w}{\partial t}+\tfrac{1}{2}\frac{\partial}{\partial x}\left(\frac{K_w}{r_w}K\frac{dp_c}{dS_w}\frac{\partial S_w}{\partial x}\right)=0. \qquad ...(2.17)$$

At this stage for definiteness of the mathematical analysis we assume a standard form of Jones (1961) for the relationship between the relative permeability-phase saturation and a linear relationship between capillary pressure-phase saturation as

$$K_w = S_w^3, K_o = S_o = 1 - \alpha_o S_w, (\alpha_o = 1.11) \quad ...(2.18)$$

$$p_c = \beta\left(S_w^{-1} - C\right), \qquad (\beta \text{ and } C \text{ are constants}). \quad ...(2.19)$$

Substituting this value in eqn. (2.17), we get

$$m\frac{\partial S_w}{\partial t} - \frac{3}{2}\frac{\beta}{r_w}K\frac{\partial}{\partial x}\left(S_w\frac{\partial S_w}{\partial x}\right) = 0. \quad ...(2.20)$$

Equation (2.20) can be converted into a dimensionless form by considering a variable

$$X = \frac{x}{L}, T = \frac{3}{2}\frac{\beta K t}{m L^2 r_w}. \quad ...(2.21)$$

Then eqn. (2.20) takes the form

$$\frac{\partial S_w}{\partial T} - \frac{\partial}{\partial X}\left(S_w\frac{\partial S_w}{\partial X}\right) = 0. \quad ...(2.22)$$

This is the nonlinear differential equation of motion for saturation.

3. Similarity Solution

The solution of eqn. (2.22) can be determined by similarity analysis method. Now eqn. (2.22) can be rewritten in the form

$$\frac{\partial S_w}{\partial T} - \left(\frac{\partial S_w}{\partial X}\right)^2 - S_w\frac{\partial^2 S_w}{\partial X^2} = 0. \quad ...(3.1)$$

To solve eqn. (3.1), we use Birkhof's technique of one parameter group transformations. Let a group T_1 consisting of a set of transformations be defined as

$$T_1 : \bar{X} = a^q X, \bar{T} = a^r T, \text{ and } \bar{S}_w = a^s S_w, \quad ...(3.2)$$

where the parameter $a \neq 0$, and q, r, s are real numbers to be determined.

Substituting eqn. (3.2) into eqn. (3.1), we obtain,

$$a^{r-s}\frac{\partial \bar{S}_w}{\partial \bar{T}} - a^{2q-2s}\left(\frac{\partial \bar{S}_w}{\partial \bar{X}}\right)^2 - a^{2q-2s}\bar{S}_w\frac{\partial^2 \bar{S}_w}{\partial \bar{X}^2} = 0. \quad ...(3.3)$$

Equation (3.3) is absolute conformally invariant under T_1, provided

$$2q - 2s = r - s. \quad ...(3.4)$$

Now, we choose to eliminate T so that the solution of eqn. (3.4) for $r \neq 0$ an equivalent to the solution of

$$2\frac{q}{r} - 2\frac{s}{r} = 1 - \frac{s}{r}. \qquad \text{...(3.5)}$$

Choosing an arbitrary constant A and then setting

$$\frac{s}{r} = A \qquad \text{...(3.6)}$$

and combining (3.6) with (3.5) we get,

$$2\frac{q}{r} = 1 + A = B$$

where B is a constant.

Thus the invariants of the group T_1 are given by

$$\eta = \frac{X}{T^{B/2}}, \; F(\eta) = S_w(X, T)/T^A. \qquad \text{...(3.7)}$$

Then eqn. (3.1) in terms of the new variable η and $F(\eta)$ is

$$T^{A-1}\left\{\left(AF(\eta) - \frac{B}{2}\eta F'(\eta)\right) - F'^2(\eta) - F(\eta) F''(\eta)\right\} = 0$$

$$\text{(since } T^{A-1} \neq 0\text{).} \qquad \text{...(3.8)}$$

Therefore,

$$FF'' + F'^2 + \frac{B}{2}\eta F' - AF = 0 \qquad \text{...(3.9)}$$

where $F = F(\eta)$.

This is nonlinear ordinary differential equation of second order.

Equation (3.9) can be solved by considering the substitution

$$F(\eta) = \eta^2 u(z) \; ; \; z = \log \eta$$

and $\quad u'(z) = p \qquad$ ...(3.10)

then eqn. (3.9) takes the form

$$upp'(u) + p^2 + \left(7u + \frac{B}{2}\right)p + u(6u + 1) = 0. \qquad \text{...(3.11)}$$

This is the Abel's equation of second kind whose solution can be obtained by considering the substitution.

$$up = \log v(z). \qquad \text{...(3.12)}$$

Substituting eqn. (3.12) into eqn. (3.11), we get

$$\frac{1}{v}\frac{dv}{du} + 7u + \frac{B}{2} + (6u + 1)\frac{u}{p} = 0. \qquad \ldots(3.13)$$

Further supposing that $\log v(z) = M(z)$, then eqn. (3.13) can take the form

$$\frac{dM}{dz} + \left(7u + \frac{B}{2}\right)\frac{M}{u} + u(6u + 1) = 0. \qquad \ldots(3.14)$$

The solution of eqn, (3.14) is

$$Me^{\int (7/\eta + (B/2F\eta))d\eta} = \int \frac{F}{\eta^3}\left(1 + \frac{6F}{\eta^2}\right) d\eta e^{\int (7/\eta + (B/2F)\eta)\, d\eta} + C \qquad \ldots(3.15)$$

where C is a constant of integration.

For evaluating the constant we can use the following boundary condition :

At $t = 0$, $\eta = \infty$ and $F(\eta) = \infty$.

Therefore, we get $C = 0$.

Then eqn. (3.15) takes the form

$$Me^{\int (7/\eta + (B/2F)\eta)\, d\eta} = \int \frac{F}{\eta^3}\left(1 + \frac{6F}{\eta^2}\right) d\eta e^{\int (7/\eta + (B/2F)\eta)\, d\eta}. \qquad \ldots(3.16)$$

Equation (3.16) gives the formal solution in terms of transcendental functions.

Conclusion

In this paper we have obtained the analytical solution of the flow of two immiscible liquids through porous media by using similarity technique and have considered that the individual pressure of the two flowing phases may be replaced by their common mean pressure (Oroveanu 1964). The mathematical formulation leads to a nonlinear differential equation which has been reduced to Abel's equation of second kind and by using a similarity solution (Hansen 1964) and its formal solution is obtained in terms of transcendental functions. We have not included any numerical illustration or graphical representations due to our particular interest in the analytical solution but the same can be easily obtained by using field values of the characteristic parameters.

Notwithstanding the limitations of the present analysis it is believed that the present similarity solution will provide useful theoretical informations of at least one complicated case of two immiscible liquid flow problems.

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