RESPONSE IN A PIEZO-ELECTRIC BAR TRANSDUCER UNDER THE INFLUENCE OF A BODY FORCE

by PRADIP KUMAR SEN, *Department of Physics, Mahadevananda Mahavidyalaya, Barrackpur, 24-Parganas (W. B.)*

(Communicated by F. C. Auluck, F.N.A.)

(*Received* 5 *March* 1973; *after revision* 5 *May* 1973)

A problem of response in a piezo-electric bar transducer under the influence of a body force has been worked out by powerful operational method due to Heaviside (1966). The transducer is vibrating in a compressional lengthwise vibration and is excited by transient stress input. The problem is discussed in three different cases, viz. (a) open circuit bar transducer, (b) resistive loading at one end of the bar and (c) mechanical signal produced by a step voltage.

INTRODUCTION

The analysis of mechanical or electrical response in a piezo-electric transducer is important from the stand point of generation and detection of ultrasonic waves. In recent years, there has been a considerable amount of studies in this branch (Ghosh and Banerjee 1965; Redwood 1961; Sinha 1962, 1967).

A rigorous theory for a crystal resonator of any form and orientation, vibrating in any desired mode, would have to take account of all possible boundary conditions, size and position of the electrodes, losses due to dielectric and mounting, nonlinear effects, coupling between different modes of vibration where the electrodes are separated by a gap from the crystal, the effects of the gap, including possible resonance effects in the air itself. While no such general theory has been attempted, special problem involving most of these considerations have been attacked by many writers. We consider a few simple cases here. In practice, the commonest types of piezo-electric transducers are the bar, vibrating compressionally lengthwise and the plate vibrating in the thickness mode. For this purpose an X-cut quartz rectangular bar is used. With this cut the compressional vibrations are produced. The experimental evidence of this mode of vibration is that acoustic waves in air are emitted from the surface. Ghosh and Bhattacharya (1971) have studied the characteristics of transient pulses, responses, etc. for a piezo-electric bar transducer, where the effect of body force has been kept out of consideration. Hence this paper discussing the problem of response in a piezo-electric bar transducer under the influence of a body force.

Explanation of the Symbols Used

X = length of the bar

Y = breadth of the bar

Z = thickness of the bar

x = variable length, measured along the length of the bar

ξ = mechanical displacement of any particle in the x-direction

ξ_0 = mechanical displacement of any particle at $x = 0$

v = velocity of propagation of wave along the bar

F = mechanical stress input to the surface normal to x-direction at $x = 0$

Q = total charge at the surface of the bar

h = piezo-electric constant

V = potential across the bar from $x = 0$ to $x = X$

E = Young's modulus of the material of the bar

C_0 = static capacitance of the bar transducer

R = resistance due to the bar transducer

ϵ = dielectric constant

D = d/dt (operator)

ρ = density of the bar

t = variable time.

SOLUTION OF THE PROBLEM

The equation of motion of a bar, vibrating compressionally lengthwise, the x-axis coinciding with the particular direction, is given by

$$\rho \frac{\partial^2 \xi}{\partial t^2} = (E - h^2\epsilon) \frac{\partial^2 \xi}{\partial x^2} + \rho H(t) \exp(-Kx) \qquad \text{...(1)}$$

where $H(t) \exp(-Kx)$ is the body force, $H(t)$ is the Heaviside unit function.

The solution of eqn. (1) is

$$\xi = A \cosh \frac{D}{v} X + B \sinh \frac{D}{v} X + \frac{\exp(-Kx)}{D^2 - v^2 K^2} \qquad \text{...(2)}$$

where A and B are arbitrary constants to be determined from initial conditions, and $v^2 = (E - h^2\epsilon)/\rho$.

The relation between end displacements and potential difference across the bar is given by

$$V = -\frac{\phi}{C_0}\{(\xi)x - (\xi)_0\} + Q/C_0, \qquad ...(3)$$

where $\phi = h\epsilon z$.

The equation for the compressional lengthwise vibration of the bar along the x-direction at $x = 0$ of the bar can be written as

$$F + \phi V = E\left(\frac{\partial \xi}{\partial x}\right)_{x=0}. \qquad ...(4)$$

ELECTRICAL RESPONSE

In order to find out the electrical response, the transducer is to be excited by transient mechanical stress input.

Two cases will be discussed here. In case I the transducer is open at both ends i.e. the transducer is cemented between two successive semi-infinite mechanical systems, and in case II to one end of the transducer there is a resistive loading analogous to the case of a fixed-free system.

Case I

Here the circuit may be taken to be open, the conditions for which are given by

$$x = 0, \quad \xi = \xi_0 \qquad ...(5)$$

$$x = X, \left(\frac{\partial \xi}{\partial x}\right) = 0 \qquad ...(6)$$

$$Q = 0. \qquad ...(7)$$

With the help of eqns. (2), (4), (5), (6) and (7) the voltage from eqn. (3) becomes

$$V\left\{1 + \frac{K_1}{D}\tanh\frac{D}{2v}X\right\} = -\frac{\phi v F}{C_0 E D}\tanh\frac{D}{2v}X$$

$$+\frac{Kv\{\exp(-KX) + 1\}}{D(D^2 - v^2K^2)}\tanh\frac{D}{2v}X - \frac{\{1 - \exp(-KX)\}}{D^2 - v^2K^2} \qquad ...(8)$$

where $K_1 = \dfrac{\phi^2 v}{C_0 E} = \dfrac{h^2\epsilon^2 Z^2 v}{C_0 E}$.

Expanding the hyperbolic tangents, using the method of partial fraction rule, the operational solution of the eqn. (8) can be written as given below, in which higher powers of K_1 are neglected, since K_1 is small.

$$V = -\frac{F}{h\epsilon Z}\left\{F_1(t) + 2K_1\sum_{n=1}^{\infty}(-1)^n f_1\left(t - \frac{nX}{v}\right)\right\} +$$

(equation contd. p. 559)

$$+ Kv\left\{\exp(-KX)+1\right\}\left\{F_2(t)+2\sum_{n=1}^{\infty}(-1)^n f_2\left(t-\frac{nX}{v}\right)\right\}$$

$$-\left\{1-\exp(-Kt)\right\}\left\{F_3(t)-2K_1\sum_{n=1}^{\infty}(-1)^n f_3\left(t-\frac{nX}{v}\right)\right\} \qquad ...(9)$$

where $F_1(t) = 1 - \exp(-K_1 t)$

$$F_2(t) = \frac{\exp(vKt)-1}{2v^2K^2(vK+K_1)} + \frac{1-\exp(-vKt)}{2v^2K^2(vK-K_1)} - \frac{1-\exp(-K_1t)}{v^2K^2K_1}$$

$$F_3(t) = \frac{1-\exp(-vKt)}{2vK(K_1-vK)} + \frac{\exp(vKt)-1}{2vK(K_1+vK)} + \frac{1-\exp(-K_1t)}{v^2K^2}$$

$$f_1(t) = t\exp(-K_1t)$$

$$f_2(t) = \frac{\exp(vKt)-1}{2v^2K^2(vK+2K_1)} + \frac{1-\exp(-vKt)}{2v^2K^2(vK-2K_1)} - \frac{1-\exp(-2K_1t)}{2v^2K^2K_1}$$

$$f_3(t) = \frac{1-\exp(-vKt)}{2v^2K^2(vK-n_1)} + \frac{\exp(vKt)-1}{2v^2K^2(vK+n_1)} + \frac{1-\exp(-n_1t)}{n_1(v^2K^2-n_1^2)}$$

$$n_1 = (n+1)K_1$$

and $\dfrac{\phi vF}{C_0EK_1} = \dfrac{F}{H\epsilon Z}$.

Case II

Here the transducer is cemented to a finite electrical impidence of resistance R ohms. The boundary conditions are :

$$x = 0,\quad \xi = \xi_0 \qquad ...(10)$$

$$x = X,\quad \xi = 0. \qquad ...(11)$$

From eqns. (3) and (11) we have

$$V = (\phi/C_0)\xi_0 + Q/C_0. \qquad ...(12)$$

The voltage V is given by

$$V = -DQR. \qquad ...(13)$$

From eqns. (2), (4), (10), (11), (12) and (13) we have

$$V\left\{1 + m/D + (K_1/D)\tanh\frac{D}{v}X\right\} = -\frac{\phi vF}{C_0ED}\tanh\frac{D}{v}X$$

$$-\frac{\exp(-KX)}{(D^2-v^2K^2)\cosh(D/v)X} - \frac{vk\tanh(D/v)X}{D(D^2-v^2K^2)} + \frac{1}{D^2-v^2K^2} \qquad ...(14)$$

where

$$\phi^2 v/C_0 E = K_1 \text{ and } m = 1/C_0 R.$$

Expanding the hyperbolic functions, using the method of partial fraction rule, the operational solution of eq. (14) can be written as given below, in which K_1 is neglected because it is small as compared to m :

$$V = -\frac{h\epsilon ZRFv}{E}\left\{f_4(t) + 2\sum_{n=1}^{\infty}(-1)^n f_4\left(t - \frac{2nX}{v}\right)\right\}$$

$$-2\exp(-KX)\sum_{n=0}^{\infty}(-1)^n f_5\left\{t - \frac{(2n+1)}{v}X\right\}$$

$$-\quad Kv\left\{f_6(t) + 2\sum_{n=1}^{\infty}(-1)^n f_6\left(t - \frac{2nX}{v}\right)\right\} + f_7(t) \qquad \text{...(15)}$$

where

$$f_4(t) = 1 - \exp(-mt)$$

$$f_5(t) = \frac{1-\exp(-vKt)}{2vK(m-vK)} + \frac{\exp(vKt)-1}{2vK(m+vK)} + \frac{1-\exp(-mt)}{v^2K^2-m^2}$$

$$f_6(t) = \frac{1-\exp(-vKt)}{2v^2K^2((vK-m)} + \frac{\exp(vKt)-1}{2v^2K^2(vK+m)} + \frac{1-\exp(-mt)}{(m^2-v^2K^2)m}$$

$$f_7(t) = \frac{1-\exp(-vKt)}{2vK(m-vK)} + \frac{\exp(vKt)-1}{2vK(m+vK)} + \frac{1-\exp(-mt)}{(v^2K^2-m^2)}$$

and

$$\phi Fv/C_0 Em = \frac{h\epsilon ZRFv}{E}.$$

Mechanical Response

In order to calculate the mechanical response, the transducer is to be excited by electrical stress input. To the end $x = 0$ the voltage V is applied which is assumed to be a step function, i.e.

$$V = 0, \quad t \leqslant 0 \qquad \text{...(16)}$$

$$V = V_0, \quad t > 0 \qquad \text{...(17)}$$

where V_0 is a constant.

It is also assumed that the transducer is rigidly backed. So the initial conditions are

$$x = 0, \quad \xi = \xi_0 \qquad \ldots(18)$$

$$x = X, \quad \xi = 0. \qquad \ldots(19)$$

From eqns. (2), (3), (18) and (19) the force is given by

$$F + \phi V = - \frac{EC_0 V_1 D}{\phi\, v} \coth (D/v)X + \frac{ED \coth (D/v)X}{v(D^2 - v^2K^2)} - \frac{ED \exp(-KX)}{v(D^2 - v^2K^2) \sinh (D/v)X} - \frac{KE}{(D^2 - v^2 K^2)} \qquad \ldots(20)$$

where $V_1 = (V - Q/C_0)$ is the resultant voltage.

Expanding the hyperbolic functions and using the method of partial fraction rule the operational solution of eqn. (20) in terms of displacement is given by

$$\xi_0 = - \frac{v(F + \phi V)}{E} \left\{ t + 2 \sum_{n=1}^{\infty} (-1)^n \left(t - \frac{2nX}{v} \right) \right\}$$

$$- \frac{2 \exp(-KX)}{vK} \sum_{n=0}^{\infty} (-1)^n \left[\cosh vK \left\{ t - \frac{(2n+1)}{v} X \right\} - 1 \right]$$

$$- \frac{1}{v^2K^2} \left[\sinh vKt - vKt + 2 \sum_{n=0}^{\infty} (-1)^n \left\{ \sinh vK \left(t - \frac{2nX}{v} \right) \right. \right.$$

$$\left. \left. - vK \left(t - \frac{2nX}{v} \right) \right\} \right] + \frac{\cosh vKt - 1}{vK}. \qquad \ldots(21)$$

Discussion

Equations (9) and (15) give the effect that the body force has on the electrical response of the bar transducer and eqn. (21) gives the effect of body force on the mechanical response of the same transducer. These equations are in agreement with those obtained by Ghosh and Bhattacharya (1971) in absence of the body force (i.e. when $k \to \infty$).

It is clear from eqn. (9) in case I, that the electrical response in a piezo-electric transducer is independent of the time constant of the circuit and governed solely by the damping constant K_1 and the constant of body force K.

The equation (15) in case II shows that the time constant of the circuit plays an important role in the electrical response of the transducer. For a very large value of time constant, the voltage developed across the grid is only due to the body force, and in absence of the body force the crystal does not vibrate at all. That is, under such circumstances the effect of body force is very important in connection with the crystal vibration. Again for a very small value of time constant, the effect of body force is negligible and the voltage developed is a function of time, representing a square wave, which is very important in connection with delay line, where the voltage remains constant over a large interval of time.

Expression (21) shows that the mechanical response ξ_0 is partly constant, partly linear and partly varies as hyperbolic sine and cosine functions of time. However, some time-delays are present with some of the time functions of eqn. (21.)

Acknowledgement

Thanks are due to Dr. R. K. Roy of Department of Physics, Burdwan University, Burdwan, for his encouragement.

References

Ghosh, S. K., and Banerjee, S. K. (1965). Responses in a piezo-electric plate transducer. *Indian J. Phys.*, **39**, 486.

Ghosh, S. K., and Bhattacharya, D. (1971). Response in a piezo-electric bar transducer. *Indian J. Physics*, **45**, 411.

Jeffreys, H., and Jeffreys, B. (1966). Methods of Mathematical Physics. Cambridge University Press.

Redwood, M. (1961). Transient performance of a piezo-electric transducer. *J. acoust. Soc. Am.*, **33**, No 4, 527.

Sinha, D. K. (1962). Note on response in a piezo-electric plate transducer with a periodic step input. *Indian J. Theor. Phys.*, **10**, No. 1, 21.

Sinha, D. K. (1967). A note on Mechanical Response in a piezo-electric ceramic transducer under the influence of a body force. *Indian J. Phys.*, **41**, 925.