GENERALIZED INFORMATION-IMPROVEMENT FUNCTIONS

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Theil (1967) has studied 'information-improvement' which has found applications in economics. A characterization of Theil's measure has been studied by Sharma and Autar (1973). In this paper a generalized measure of 'information-improvement' through a functional equation has been obtained.

1. INTRODUCTION

Let $P = (p_1, \ldots, p_n)$ be the posterior probability distribution of a set of n events on the basis of an experiment E whose a priori distribution (prediction) is $Q = (q_1, \ldots, q_n)$, then

$$I(P; Q) = \sum_{i=1}^{n} p_i \log_2 (p_i/q_i),\ p_i, q_i \geqslant 0,\ \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1 \qquad \ldots(1.1)$$

is Kullback's (1959) expected information from this experiment. Theil (1967) has given two important interpretations of this measure : one as the information content of the prediction and other as information inaccuracy of the prediction. Continuing the second interpretation Theil defines 'information-improvement measure'

$$T(P; Q; R) = \sum_{i=1}^{n} p_i \log_2 (r_i/q_i) \qquad \ldots(1.2)$$

where $R = (r_1, \ldots, r_n)$, $r_i \geqslant 0$, $\sum_{i=1}^{n} r_i = 1$ is the revised prediction probability distribution of the original prediction $Q = (q_1, \ldots, q_n)$ on the basis of the distribution $P = (p_1, \ldots, p_n)$ obtained from the experiment E.

A systematic study through functional equation which arises by considering certain postulates of this measure is made in a paper (for entropy refer Daroczy 1970) by Sharma and Autar (1973) defining the information-improvement function as follows :

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A function $f(x; y; z)$ satisfying the functional equation

$$f(x_1; y_1; z_1) + (1 - x_1) f\left(\frac{x_2}{1 - x_1}; \frac{y_2}{1 - y_1}; \frac{z_2}{1 - z_1}\right)$$
$$= f(x_2; y_2; z_2) + (1 - x_2) f\left(\frac{x_1}{1 - x_2}; \frac{y_1}{1 - y_2}; \frac{z_1}{1 - z_2}\right) \quad ...(1.3)$$

for all (x_1, x_2), (y_1, y_2) and $(z_1, z_2) \in D$ where

$$D = \{(t_1, t_2) : t_1, t_2 \in [0, 1), t_1 + t_2 \leqslant 1\} \quad ...(1.4)$$

and the boundary conditions

$$f(0; 0; 0) = f(1; 1; 1); f(1; \tfrac{1}{2}; 1) = 1 \quad ...(1.5)$$

is called information-improvement function which under some additional conditions like continuity etc. leads to Theil's information-improvement function

$$f_T(x; y; z) = x \log_2 (z/y) + (1 - x) \log_2 \{(1 - z)(1 - y)^{-1}\} \quad ...(1.6)$$

if $x, y, z \in [0, 1]$ excluding those points of the region where function becomes $\pm \infty$. $\left(\text{We assume } 0 \log 0 = 0 \log \frac{0}{0} = 0\right)$.

In this paper we have generalized (1.3) for two parameters viz. $\alpha, \beta > 0$. A characterization of the corresponding generalized information-improvement function and measure is given. Some of the properties of the generalized measure have also been incorporated.

2. Information-Improvement Function of Type (α, β) and a Characterization Theorem

We generalize (1.3) by introducing two parameters as

$$f(x_1; y_1; z_1) + (1 - x_1)^{\alpha} (1 - y_1)^{\alpha-\beta} (1 - z_1)^{\beta-\alpha} f\left(\frac{x_2}{1 - x_1}; \frac{y_2}{1 - y_1}; \frac{z_2}{1 - z_1}\right)$$
$$= f(x_2; y_2; z_2) + (1 - x_2)^{\alpha} (1 - y_2)^{\alpha-\beta} (1 - z_2)^{\beta-\alpha}$$
$$\times f\left(\frac{x_1}{1 - x_2}; \frac{y_1}{1 - y_2}; \frac{z_1}{1 - z_2}\right) \quad ...(2.1)$$

$\alpha, \beta > 0$ with $\beta \neq \alpha$.

Definition 1 — A real function $f : K \to R$ where $K = [0, 1] \times [0, 1] \times [0, 1]$ and R, set of reals, is called information-improvement function of type (α, β) if f satisfies (2.1) and (1.5).

Now we characterize information-improvement function of type (α, β).

Theorem 1 — If $f(x; y; z)$ satisfies the functional equation (2.1) and the boundary conditions (1.5) then for all $x, y, z \in [0, 1]$

$$f(x; y; z) = f_T^{(\alpha, \beta)}(x; y; z)$$

where

$$f_T^{(\alpha, \beta)}(x; y; z) = (2^{\beta-\alpha} - 1)^{-1} [x^{\alpha} y^{\alpha-\beta} z^{\beta-\alpha} + (1-x)^{\alpha}(1-y)^{\alpha-\beta}(1-z)^{\beta-\alpha} - 1] \quad \text{...(2.2)}$$

for $\alpha, \beta > 0$ with $\beta \neq \alpha$.

PROOF: By putting $x_1 = y_1 = z_1 = 0$ in (2.1) and using (1.5) we get

$$f(0; 0; 0) = f(1; 1; 1) = 0. \quad \text{...(2.3)}$$

Taking $x_2 = 1 - x_1$, $y_2 = 1 - y_1$ and $z_2 = 1 - z_1$ in (2.1) and utilizing (2.3) we have

$$f(x_1; y_1; z_1) = f(1 - x_1; 1 - y_1; 1 - z_1), \text{ for all } x_1, y_1, z_1 \in [0, 1]. \quad \text{...(2.4)}$$

Next setting $p_1 = 1 - x_1$, $q_1 = 1 - y_1$, $r_1 = 1 - z_1$, $p_2 = x_2 (1 - x_1)^{-1}$, $q_2 = y_2 (1 - y_1)^{-1}$ and $r_2 = z_2(1 - z_1)^{-1}$ in (2.1) and using (2.4), we get

$$f(p_1; q_1; r_1) + p_1^{\alpha} q_1^{\alpha-\beta} r_1^{\beta-\alpha} f(p_2; q_2; r_2)$$

$$= f(p_1 p_2; q_1 q_2; r_1 r_2) + (1 - p_1 p_2)^{\alpha} (1 - q_1 q_2)^{\alpha-\beta} (1 - r_1 r_2)^{\beta-\alpha}$$

$$\times f\left(\frac{1 - p_1}{1 - p_1 p_2}; \frac{1 - q_1}{1 - q_1 q_2}; \frac{1 - r_1}{1 - r_1 r_2}\right). \quad \text{...(2.5)}$$

Consider now for arbitrary $p_1, q_1, r_1 \in (0, 1]$, $p_2, q_2, r_2 \in [0, 1]$ the function F defined by

$$F(p_1, p_2; q_1, q_2; r_1, r_2) = f(p_1; q_1; r_1)$$

$$+ \left[p_1^{\alpha} q_1^{\alpha-\beta} r_1^{\beta-\alpha} + (1 - p_1)^{\alpha} (1 - q_1)^{\alpha-\beta} (1 - r_1)^{\beta-\alpha}\right] f(p_2; q_2; r_2). \quad \text{...(2.6)}$$

We prove that F defined above is symmetric, i.e.

$$F(p_1, p_2; q_1, q_2; r_1, r_2) = F(p_2, p_1; q_2, q_1; r_2, r_1). \quad \text{...(2.7)}$$

Now (2.6) with (2.5) gives

$$F(p_1, p_2; q_1, q_2; r_1, r_2)$$

$$= f(p_1 p_2; q_1 q_2; r_1 r_2) + (1 - p_1 p_2)^{\alpha} (1 - q_1 q_2)^{\alpha-\beta} (1 - r_1 r_2)^{\beta-\alpha}$$

$$\times \left[f\left(\frac{1 - p_1}{1 - p_1 p_2}; \frac{1 - q_1}{1 - q_1 q_2}; \frac{1 - r_1}{1 - r_1 r_2}\right) + \left(\frac{1 - p_1}{1 - p_1 p_2}\right)^{\alpha}\right.$$

$$\left. \times \left(\frac{1 - q_1}{1 - q_1 q_2}\right)^{\alpha-\beta} \left(\frac{1 - r_1}{1 - r_1 r_2}\right)^{\beta-\alpha} f(p_2; q_2; r_2)\right]. \quad \text{...(2.8)}$$

If we set $p_1^* = (1-p_1)(1-p_1p_2)^{-1}$, $q_1^* = (1-q_1)(1-q_1q_2)^{-1}$ and $r_1^* = (1-r_1)(1-r_1r_2)^{-1}$, then from (2.4) and (2.5) it can be easily verified that

$$\begin{aligned}
& G(p_1, p_2; q_1, q_2; r_1, r_2) \\
&= f\left(\frac{1-p_1}{1-p_1p_2}; \frac{1-q_1}{1-q_1q_2}; \frac{1-r_1}{1-r_1r_2}\right) \\
&\quad + \left(\frac{1-p_1}{1-p_1p_2}\right)^{\alpha}\left(\frac{1-q_1}{1-q_1q_2}\right)^{\alpha-\beta}\left(\frac{1-r_1}{1-r_1r_2}\right)^{\beta-\alpha} f(p_2; q_2; r_2) \\
&= f\left(\frac{1-p_2}{1-p_1p_2}; \frac{1-q_2}{1-q_1q_2}; \frac{1-r_2}{1-r_1r_2}\right) \\
&\quad + \left(\frac{1-p_2}{1-p_1p_2}\right)^{\alpha}\left(\frac{1-q_2}{1-q_1q_2}\right)^{\alpha-\beta}\left(\frac{1-r_2}{1-r_1r_2}\right)^{\beta-\alpha} f(p_1; q_1; r_1) \\
&= G(p_2, p_1; q_2, q_1; r_2, r_1).
\end{aligned}$$

This proves (2.7). Now putting $p_2 = r_2 = 0$, $q_2 = \frac{1}{2}$ in (2.7) then from (2.6), (2.4) and $f(1; \frac{1}{2}; 1) = 1$ we get

$$f(p_1; q_1; r_1) = (2^{\beta-\alpha} - 1)^{-1}\Big[p_1^{\alpha} q_1^{\alpha-\beta} r_1^{\beta-\alpha} + (1-p_1)^{\alpha}(1-q_1)^{\alpha-\beta} \times (1-r_1)^{\beta-\alpha} - 1\Big]$$

for all $p_1; q_1; r_1 \in (0, 1]$, $\beta \neq \alpha$. This formula is true in the cases $p_1 = q_1 = r_1 = 0$ or $p_1 = q_1 = r_1 = 1$ by the boundary conditions (1.5). Also (2.7) with (2.6), (2.4) and the above relation, we can easily determine the values of $f(1; 1; 0)$, $f(0; 0; 1)$, $f(1; 0; 0)$, $f(0; 1; 1)$, $f(1; 0; 1)$, $f(0; 1; 0)$ and $f(1; q_2; 0)$, $f(1; 0; r_2)$, $f(p_2; 0; 1) \in [0, 1)$ etc. and value of each is $(1 - 2^{\beta-\alpha})^{-1}$ This completes the proof of the theorem.

Note: It is to be noted that for $\alpha = 1$, $\beta \to 1$, (2.2) reduces to Theil's information-improvement function (1.6).

Definition 2 — Let $P = (p_1, ..., p_n)$, $Q = (q_1, ..., q_n)$ and $R = (r_1, ..., r_n)$ be three probability distributions belonging to $\Delta_n = \{(p_1, ..., p_n) : p_i \geqslant 0 \sum_{i=1}^{n} p_i = 1\}$.

We define $I_n^{(\alpha,\beta)}(P; Q; R)$, the information-improvement of type (α, β) as

$$I_n^{(\alpha,\beta)}(P; Q; R) = \sum_{i=2}^{n} s_i^{\alpha} t_i^{\alpha-\beta} u_i^{\beta-\alpha} f\left(\frac{p_i}{s_i}; \frac{q_i}{t_i}; \frac{r_i}{u_i}\right) \qquad ...(2.9)$$

where $s_i = p_1 + ... + p_i$, $t_i = q_1 + ... + q_i$, $u_i = r_1 + ... + r_i$ $(i = 2, 3, ..., n)$ and f is an information-improvement function of type (α, β). Now, in view of (2.2), (2.9) reduces to

$$I_n^{(\alpha,\beta)}(P;Q;R) = (2^{\beta-\alpha}-1)^{-1}\left(\sum_{i=1}^{n} p_i^{\alpha}\, q_i^{\alpha-\beta}\, r_i^{\beta-\alpha} - 1\right),\ \beta \neq \alpha. \quad \text{...(2.10)}$$

Note : $I_n^{(\alpha,\beta)}(P;Q;R)$ in (2.10) is a generalization of Theil's information-improvement as for $\alpha = 1$ and $\beta \to 1$, (2.10) reduces to (1.2).

3. Properties and Characterization of $I_n^{(\alpha,\beta)}(P;Q;R)$

This section deals with some properties and a characterization of $I_n^{(\alpha,\beta)}(P;Q;R)$. It is the characterization contained in Theorem 5 that places the new measure on a sound mathematical basis and the properties discussed make it amenable for further study and applications. The characterization has been done on the basis of certain simple postulates. As regards some of the properties contained in various theorems, Theorem 2 establishes a relation between two sets of measures to the measure associated with the direct product of their distributions. This is a property which is basic and brings out the non-additive character of the measure studied in this paper. If $\alpha = 1$, $\beta \to 1$, the additive property tallies with that of Theil's measure. Theorem 3 deals with a kind of grouping property and Theorem 4 with what may be called as convexity of the measure. It may be pointed out that such properties are useful while optimizing a given measure which may be an area for future study of this measure.

Several properties like symmetry, expansibility i.e.

$$I_n^{(\alpha,\beta)}\begin{pmatrix} p_1, \ldots, p_n, 0; \\ q_1, \ldots, q_n, 0; \\ r_1, \ldots, r_n, 0 \end{pmatrix} = I_n^{(\alpha,\beta)}\begin{pmatrix} p_1, \ldots, p_n; \\ q_1, \ldots, q_n; \\ r_1, \ldots, r_n \end{pmatrix}$$

and normalization $\left(I_2^{(\alpha,\beta)}(1, 0; \tfrac{1}{2}; \tfrac{1}{2}; 1, 0) = 1 \right)$ etc. (refer Aczél 1969) can be easily derived for $I_n^{(\alpha,\beta)}(P;Q;R)$. We state now some other properties.

(*i*) *Strongly Additive Type* (α, β)

$$I_{mn}^{(\alpha,\beta)}\begin{pmatrix} p_1p_{11}, p_1p_{12}, \ldots, p_1p_{1n}, \ldots, p_mp_{m1}, p_mp_{m2}, \ldots, p_mp_{mn}; \\ q_1q_{11}, q_1q_{12}, \ldots, q_1q_{1n}, \ldots, q_mq_{m1}, q_mq_{m2}, \ldots, q_mq_{mn} \\ r_1r_{11}, r_1r_{12}, \ldots, r_1r_{1n}, \ldots, r_mr_{m1}, r_mr_{m2}, \ldots, r_mr_{mn} \end{pmatrix} =$$

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$$= I_m^{(\alpha,\beta)}\begin{pmatrix} p_1, \ldots, p_m; \\ q_1, \ldots, q_m; \\ r_1, \ldots, r_m \end{pmatrix} + \sum_{j=1}^{m} p_j^{\alpha} q_j^{\alpha-\beta} r_j^{\beta-\alpha} I_n^{(\alpha,\beta)}\begin{pmatrix} p_{j1}, \ldots, p_{jn}; \\ q_{j1}, \ldots, q_{jn}; \\ r_{j1}, \ldots, r_{jn} \end{pmatrix}$$

where $\sum_{k=1}^{n} p_{jk} = \sum_{k=1}^{n} q_{jk} = \sum_{k=1}^{n} r_{jk} = 1$ for all $j = 1, 2, \ldots, m$.

(ii) *Recursive Type* (α, β)

$$I_n^{(\alpha,\beta)}\begin{pmatrix} p_1, p_2, \ldots, p_n; \\ q_1, q_2, \ldots, q_n; \\ r_1, r_2, \ldots, r_n \end{pmatrix} - I_{n-1}^{(\alpha,\beta)}\begin{pmatrix} p_1 + p_2, p_3, \ldots, p_n; \\ q_1 + q_2, q_3, \ldots, q_n; \\ r_1 + r_2, r_3, \ldots, r_n \end{pmatrix}$$

$$= (p_1 + p_2)^{\alpha} (q_1 + q_2)^{\alpha-\beta} (r_1 + r_2)^{\beta-\alpha} I_2^{(\alpha,\beta)}\begin{pmatrix} p_1/(p_1+p_2), p_2/(p_1+p_2); \\ q_1/(q_1+q_2), q_2/(q_1+q_2); \\ r_1/(r_1+r_2), r_2/(r_1+r_2) \end{pmatrix}$$

for all $(p_1, \ldots, p_n)$, and $(q_1, \ldots, q_n)$ and $(r_1, \ldots, r_n) \in \triangle_n$.

An interesting special case of (1) above is given in the next theorem.

Theorem 2 — Let $P_1 = (p_{11}, p_{12}, \ldots, p_{1n}) \in \triangle_n$, $P_2 = (p_{21}, p_{22}, \ldots, p_{2m}) \in \triangle_m$ and similar notations for Q_1, Q_2 and R_1, R_2. Define

$$P_1 * P_2 = (p_{11}\, p_{21}, \ldots, p_{11}\, p_{2m}, \ldots, p_{1n}\, p_{21}, \ldots, p_{1n}\, p_{2m}) \qquad \ldots(3.1)$$

then we have

$$I_{mn}^{(\alpha,\beta)}(P_1 * P_2; Q_1 * Q_2; R_1 * R_2) = I_n^{(\alpha,\beta)}(P_1; Q_1; R_1) + I_m^{(\alpha,\beta)}(P_2; Q_2; R_2)$$

$$+ (2^{\beta-\alpha} - 1)\, I_n^{(\alpha,\beta)}(P_1; Q_1; R_1)\, I_m^{(\alpha,\beta)}(P_2; Q_2; R_2) \qquad \ldots(3.2)$$

for $\alpha, \beta > 0$ $(\beta \neq \alpha)$.

PROOF: We have

$$I_{mn}^{(\alpha,\beta)}(P_1 * P_2;\; Q_1 * Q_2; R_1 * R_2)$$

$$= (2^{\beta-\alpha} - 1)^{-1} \left[\sum_{i=1}^{n} \sum_{j=1}^{m} (p_{1i}\, p_{2j})^{\alpha} (q_{1i}\, q_{2j})^{\alpha-\beta} (r_{1i}\, r_{2j})^{\beta-\alpha} - 1\right]$$

$$= (2^{\beta-\alpha} - 1)^{-1} \left[\sum_{i=1}^{n}\left(p_{1i}^{\alpha}\, q_{1i}^{\alpha-\beta}\, r_{1i}^{\beta-\alpha}\right) \sum_{j=1}^{m}\left(p_{2j}^{\alpha}\, q_{2j}^{\alpha-\beta}\, r_{2j}^{\beta-\alpha}\right) - 1\right]$$

$$= (2^{\beta-\alpha} - 1)^{-1} \Big[\Big\{ (2^{\beta-\alpha} - 1)\, I_n^{(\alpha,\beta)}(P_1; Q_1; R_1) + 1 \Big\} \times \Big\{ (2^{\beta-\alpha} - 1)\, I_m^{(\alpha,\beta)}(P_2; Q_2; R_2) + 1 \Big\} - 1 \Big]$$

$$= I_n^{(\alpha,\beta)}(P_1; Q_1; R_1) + I_m^{(\alpha,\beta)}(P_2; Q_2; R_2) + (2^{\beta-\alpha} - 1)\, I_n^{(\alpha,\beta)}(P_1; Q_1; R_1)\, I_m^{(\alpha,\beta)}(P_2; Q_2; R_2).$$

When $\beta \to 1$ and $\alpha = 1$ the last term in (3.2) vanishes and we have

$$I_{mn}^{(\alpha,\beta)}(P_1 * P_2; Q_1 * Q_2; R_1 * R_2) = I_n^{(\alpha,\beta)}(P_1; Q_1; R_1) + I_m^{(\alpha,\beta)}(P_2; Q_2; R_2).$$

Also for $\beta \neq \alpha$ we have

$$I_{mn}^{(\alpha,\beta)}(P_1 * P_2; Q_1 * Q_2; R_1 * R_2) \gtrless I_n^{(\alpha,\beta)}(P_1 : Q_1 : R_1) + I_m^{(\alpha,\beta)}(P_2; Q_2; R_2)$$

according as

$$(2^{\beta-\alpha} - 1)\, I_n^{(\alpha,\beta)}(P_1; Q_1; R_1)\, I_m^{(\alpha,\beta)}(P_2; Q_2; R_2) \gtrless 0.$$

The following theorem is a generalization of recursitivity of $I_n^{(\alpha,\beta)}$.

Theorem 3 — Let α, β be the positive numbers such that $\beta \neq \alpha$. For $n \geqslant k + 1$ where $k \geqslant 2$,

$$I_n^{(\alpha,\beta)}\begin{pmatrix} p_1, \ldots, p_n; \\ q_1, \ldots, q_n; \\ r_1, \ldots, r_n \end{pmatrix} - I_{n-k+1}^{(\alpha,\beta)}\begin{pmatrix} \Sigma\, p_i, p_{k+1}, \ldots, p_n; \\ \Sigma\, q_i, q_{k+1}, \ldots, q_n; \\ \Sigma\, r_i, r_{k+1}, \ldots, r_n \end{pmatrix}$$

$$= (\Sigma\, p_i)^{\alpha} (\Sigma\, q_i)^{\alpha-\beta} (\Sigma\, r_i)^{\beta-\alpha}\, I_k^{(\alpha,\beta)}\begin{pmatrix} p_1/\Sigma\, p_i, \ldots, p_k/\Sigma\, p_i; \\ q_1/\Sigma\, q_i, \ldots, q_k/\Sigma\, q_i; \\ r_1/\Sigma\, r_i, \ldots, r_k/\Sigma\, r_i \end{pmatrix} \quad \ldots(3.3)$$

for all $(p_1, \ldots, p_n)$, $(q_1, \ldots, q_n)$ and $(r_1, \ldots, r_n) \in \Delta_n$ and Σ stands for $\sum_1^k$ unless otherwise specified.

PROOF : Left-hand expression of (3.3) can be written as

$$(2^{\beta-\alpha}-1)^{-1}\Big[\Big(\sum_{i=1}^{n} p_i^{\alpha} q_i^{\alpha-\beta} r_i^{\beta-\alpha}-1\Big)-\Big\{(\Sigma p_i)^{\alpha}(\Sigma q_i)^{\alpha-\beta}(\Sigma r_i)^{\beta-\alpha}+\sum_{j=k+1}^{n} p_j^{\alpha} q_j^{\alpha-\beta} r_j^{\beta-\alpha}-1\Big\}\Big]$$

$$=(\Sigma p_i)^{\alpha}(\Sigma q_i)^{\alpha-\beta}(\Sigma r_i)^{\beta-\alpha}(2^{\beta-\alpha}-1)^{-1}\times\Big[\sum_{j=1}^{k}(p_j/\Sigma p_i)^{\alpha}(q_j/\Sigma q_i)^{\alpha-\beta}(r_j/\Sigma r_i)^{\beta-\alpha}-1\Big]$$

$$=(\Sigma p_i)^{\alpha}(\Sigma q_i)^{\alpha-\beta}(\Sigma r_i)^{\beta-\alpha} I_k^{(\alpha,\beta)}\begin{pmatrix} p_1/\Sigma p_i, \ldots, p_k/\Sigma p_i; \\ q_1/\Sigma q_i, \ldots, q_k/\Sigma q_i; \\ r_1/\Sigma r_i, \ldots, r_k/\Sigma r_i \end{pmatrix}.$$

It is an open problem as to for what values of α and β, $I_n^{(\alpha,\beta)} > 0$. Considering the possibility of these being positive we may prove the following theorem.

Theorem 4 — For $(p_1, \ldots, p_n) \in \Delta_n$, $(q_1, \ldots, q_m)$, $(r_1, \ldots, r_m)$ and $(q_{1i}, \ldots, q_{mi}) \in \Delta_m$ $(i = 1, \ldots, n)$ we have

$$I_m^{(\alpha,\beta)}\begin{pmatrix} \sum_{i=1}^{n} p_i q_{1i}, \ldots, \sum_{i=1}^{n} p_i q_{mi}; \\ q_1, \ldots, q_m; \\ r_1, \ldots, r_m \end{pmatrix} \geqslant \sum_{i=1}^{n} p_i\, I_m^{(\alpha,\beta)}\begin{pmatrix} q_{1i}, \ldots, q_{mi}; \\ q_1, \ldots, q_m; \\ r_1, \ldots, r_m \end{pmatrix} \quad \ldots(3.4)$$

where $0 < \alpha < 1$, $\beta > 0$. The inequality is reversed if $1 < \alpha < \infty$.

PROOF : We have

$$I_m^{(\alpha,\beta)}\begin{pmatrix} \sum_{i=1}^{n} p_i q_{1i}, \ldots, \sum_{i=1}^{n} p_i q_{mi}; \\ q_1, \ldots, q_m; \\ r_1, \ldots, r_m \end{pmatrix} = (2^{\beta-\alpha}-1)^{-1}\Big[\sum_{k=1}^{m}\Big(\sum_{i=1}^{n} p_i q_{ki}\Big)^{\alpha} q_k^{\alpha-\beta} r_k^{\beta-\alpha}-1\Big]$$

$$\geqslant (2^{\beta-\alpha}-1)^{-1}\left(\sum_{k=1}^{m}\sum_{i=1}^{n} p_i\, q_{ki}^{\alpha}\, q_k^{\alpha-\beta}\, r_k^{\beta-\alpha} - 1\right)$$

if $\alpha < 1$ (Gallager 1968)

$$= \sum_{i=1}^{n} p_i\, I_m^{(\alpha,\beta)}\begin{pmatrix} q_{1i}, \ldots, q_{mi}; \\ q_1, \ldots, q_m; \\ r_1, \ldots, r_m \end{pmatrix}.$$

Since $\left(\sum_{i=1}^{n} p_i\, q_{ki}\right)^{\alpha} \leqslant \sum_{i=1}^{n} p_i\, q_{ki}^{\alpha}$ for $\alpha > 1$ (Gallager 1968), therefore, inequality in (3.4) is reversed.

In the next theorem we attempt to characterize $I_n^{(\alpha,\beta)}(P; Q; R)$ by imposing some conditions over a sequence of functions K_n; $S_n \to R$, $S_n = \triangle_n \times \triangle_n \times \triangle_n$ $(n = 2, 3, \ldots)$ so that

$$K_n(P; Q; R) = I_n^{(\alpha,\beta)}(P; Q; R) \qquad \ldots(3.5)$$

for $\alpha, \beta > 0$ $(\beta \neq \alpha)$.

Theorem 5 — Let $K_n : S_n \to R(n = 2, 3, \ldots)$ be a sequence of mappings satisfying the following postulates :

(I) $K_3\begin{pmatrix} p_1, p_2, p_3; \\ q_1, q_2, q_3; \\ r_1, r_2, r_3 \end{pmatrix}$ is symmetric function of its variables

$\left\{\begin{matrix} p_i \\ q_i \\ r_i \end{matrix}\right\}$, $i = 1, 2, 3$ $(p_1 + p_2 + p_3 = 1$ etc).

(II) $K_2(1, 0; \frac{1}{2}, \frac{1}{2}; 1, 0) = 1$.

(III) $K_n\begin{pmatrix} p_1, p_2, \ldots, p_n; \\ q_1, q_2, \ldots, q_n; \\ r_1, r_2, \ldots, r_n \end{pmatrix} - K_{n-1}\begin{pmatrix} p_1+p_2, p_3, \ldots, p_n; \\ q_1+q_2, q_3, \ldots, q_n; \\ r_1+r_2, r_3, \ldots, r_n \end{pmatrix} =$

(*equation contd. p.* 562)

$$= (p_1 + p_2)^{\alpha} (q_1 + q_2)^{\alpha-\beta} (r_1 + r_2)^{\beta-\alpha} K_2 \begin{pmatrix} p_1/(p_1 + p_2), p_2/(p_1 + p_2); \\ q_1/(q_1 + q_2), q_2/(q_1 + q_2); \\ r_1/(r_1 + r_2), r_2/(r_1 + r_2) \end{pmatrix}$$

for all $p_i, q_i, r_i \geqslant 0$ with $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} r_i = 1$ ($n = 3, 4, \ldots; p_1 + p_2 > 0$, $q_1 + q_2 > 0$ and $r_1 + r_2 > 0$).

Then the function f defined by

$$f(x; y; z) = K_2(x, 1 - x; y, 1 - y; z, 1 - z), \; x, y, z \in [0, 1] \qquad \ldots(3.6)$$

is an information-improvement function of type (α, β) and that (3.5) holds.

PROOF : For arbitrary (x_1, x_2), (y_1, y_2) and $(z_1, z_2) \in D$ we have from postulate I,

$$K_3 \begin{pmatrix} x_1, 1 - x_2 - x_1, x_2; \\ y_1, 1 - y_2 - y_1, y_2; \\ z_1, 1 - z_2 - z_1, z_2 \end{pmatrix} = K_3 \begin{pmatrix} x_2, 1 - x_1 - x_2, x_1; \\ y_2, 1 - y_1 - y_2, y_1; \\ z_2, 1 - z_1 - z_2, z_1 \end{pmatrix}$$

from which it follows by (III) and (3.6) that

$$f(1 - x_1; 1 - y_1; 1 - z_1) + (1 - x_1)^{\alpha} (1 - y_1)^{\alpha-\beta} (1 - z_1)^{\beta-\alpha} \times f\left(\frac{x_2}{1 - x_1}; \frac{y_2}{1 - y_1}; \frac{z_2}{1 - z_1}\right) = f(1 - x_2; 1 - y_2; 1 - z_2) + (1 - x_2)^{\alpha} (1 - y_2)^{\alpha-\beta} (1 - z_2)^{\beta-\alpha} f\left(\frac{x_1}{1 - x_2}; \frac{y_1}{1 - y_2}; \frac{z_1}{1 - z_2}\right). \qquad \ldots(3.7)$$

Now putting $x_1 = y_1 = z_1 = 0$ and $x_2 = y_2 = z_2 = \frac{1}{2}$ in (3.7) we get

$$f(1; 1; 1) = 2^{-\alpha} f(0; 0; 0). \qquad \ldots(3.8)$$

Again from postulate (I) and (3.6) we have

$$K_3 \begin{pmatrix} 1, 0, 0; \\ 1, 0, 0; \\ 1, 0, 0 \end{pmatrix} = K_3 \begin{pmatrix} 0, 1, 0; \\ 0, 1, 0; \\ 0, 1, 0 \end{pmatrix}$$

or $\quad f(1; 1; 1) = f(0; 0; 0). \qquad \ldots(3.9)$

Combining (3.9) with (3.8) we get

$$f(1; 1; 1) = f(0; 0; 0) = 0. \qquad \ldots(3.10)$$

Next, from postulates (III) (for $n = 3$), postulate (I) and (3.6) we have

$$f(x_1; y_1; z_1) = f(1 - x_1; 1 - y_1; 1 - z_1) \quad \text{...(3.11)}$$

for all $x_1, y_1, z_1 \in [0, 1]$.

Now (3.7) with (3.11) gives (2.1). Thus f defined in (3.6) is an information-improvement function of type (α, β). Further we have from Theorem 1,

$$f(x; y; z) = f_T^{(\alpha,\beta)}(x; y; z).$$

Now result (3.5) follows from the repeated application of postulate (III) using (3.6).

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