OSCILLATORY FREE CONVECTION FROM AN INFINITE HORIZONTAL CYLINDER IN SECOND-ORDER FLUIDS

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Oscillatory free convection from an infinite horizontal cylinder in a second-order fluid has been discussed. The flow equations are formulated by using Blasius coordinates. The purely oscillatory temperature distribution of the cylinder induces a steady state heat transfer rate from it which increases with the increase of the non-Newtonian parameter β and with the decrease of the Prandtl number σ.

1. INTRODUCTION

The constitutive equation of an incompressible second-order fluid has been suggested by Coleman and Noll (1960) as

$$P_{ij} = - p\delta_{ij} + \mu_1 A_{(1)ij} + \mu_2 A_{(2)ij} + \mu_3 A_{(1)ik} A_{(1)j}^{\;k} \qquad \ldots(1)$$

where

$$A_{(1)ij} = u_{i,j} + u_{j,i}$$

$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2\, u_{m,i}\, u^{m}_{,j} \qquad \ldots(2)$$

P_{ij} is the stress tensor; u_i, a_i the velocity and the acceleration vectors respectively; μ_1, μ_2, μ_3 the material constants; p the indeterminate hydrostatic pressure; and δ_{ij} the Kronecker delta. A comma denotes a covariant differentiation with respect to the symbol following it.

In this paper we have considered the effect of second order terms in the constitutive equation on oscillatory free convection from an infinite horizontal cylinder. The temperature T_w on the cylinder immersed in a second-order fluid is given by

$$T_w = T_0 (1 + a \cos \omega t)$$

where T_0 is the temperature at large distances from the cylinder and t is time As in all problems of free convection there is no obvious typical velocity scale and we find that the appropriate velocity scale in this case is $g\alpha a T_0/\omega$. The frequency of oscillation ω is so large that the frequency parameter ϵ $(= g\alpha a T_0/\omega^2 R)$ is small, where g is the acceleration due to gravity, α the coefficient of thermal expansion and R the typical radius of curvature of the cylinder. It is seen that the solution does not satisfy the boundary conditions at large distances from the cylinder, nevertheless it gives a correct picture for the flow very near the cylinder. The corresponding problem for a Newtonian fluid has been solved by Merkin (1967) who has also developed an outer layer solution that matches the inner layer solution and satisfies the boundary conditions at infinity.

2. Equations of Motion

The fluid is assumed to be almost incompressible, so that changes in density are important only in producing buoyancy forces. The material constants μ_1, μ_2, μ_3 and the thermometric conductivity K are taken as constants and in the energy equation the work done by the pressure and the effects of viscous dissipation are neglected.

The infinite cylinder is fixed with the axis horizontal, and so the problem is essentially two dimensional. The coordinate x is defined as the distance measured along the surface of the cylinder, the lowest point being the origin $x = 0$; and the coordinate y is defined to be the distance measured normally outwards from the cylinder. The angle θ is taken to be the angle made by the outward normal with the downward vertical. With the coordinate system specified above, the boundary layer equations have been deduced by Srivastava (1966, 1967) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots(3)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\alpha(T - T_0)\sin\theta + \nu_1\frac{\partial^2 u}{\partial y^2} + \nu_2\left[\frac{\partial^3 u}{\partial t\partial y^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\cdot\frac{\partial^2 v}{\partial y^2} + u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right] \qquad \ldots(4)$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = K\frac{\partial^2 T}{\partial y^2}, \qquad \ldots(5)$$

where u, v are the velocity components in the directions of x and y respectively within the boundary layer region, T is the temperature of the fluid, and $\nu_1 = \mu_1/\rho$, $\nu_2 = \mu_2/\rho$, ρ is the density of the fluid. The eqn. (4) is independent of the material constant μ_3 but the hydrostatic pressure p is modified and is given by

$$P = p - (2\mu_2 + \mu_3)\left(\frac{\partial u}{\partial y}\right)^2 \qquad \ldots(6)$$

If desired, the pressure P can be obtained from the equation

$$\frac{u^2}{R} = \frac{1}{\rho}\frac{\partial P}{\partial y} + g\alpha(T - T_0)\cos\theta. \qquad ...(7)$$

The appropriate Reynolds number of the boundary layer is $R_e = (g\alpha a T_0 R)/\omega\nu_1$. In these boundary layer equations the largest terms neglected are $O[(R_e)^{1/2}]$. We also suppose that $\frac{\mu_2\omega}{\mu_1} \sim O(1)$, $\frac{\mu_3\omega}{\mu_1} \sim O(1)$. The conditions imposed on the curvature κ_1 of the cylinder to make boundary layer simplifications are

$$\kappa_1\delta << 1, \; R\delta(d\kappa_1/dx) << 1,$$

where δ is a measure of the boundary layer thickness, and is $O\,[(K/\omega)^{1/2}]$.

The boundary coditions are

$$u = v = 0, \quad T - T_0 = aT_0\cos\omega t, \quad \text{on } y = 0$$

$$u \to 0, \quad T \to T_0 \text{ as } y \to \infty. \qquad ...(8)$$

3. Solution of the Equations

From the continuity equation (3) we can define a stream function ψ by

$$u = \frac{\partial\psi}{\partial y},\; v = -\frac{\partial\psi}{\partial x} \qquad ...(9)$$

We introduce nondimensional variables ξ, τ, η, G and F as

$$\xi = x/R, \quad \tau = \omega t, \quad \eta = (\omega/2K)^{1/2}\,y \qquad ...(10)$$

$$(T - T_0)/aT_0 = G(\xi, \eta, \tau) \qquad ...(11)$$

$$\psi = \left(\frac{2K}{\omega}\right)^{\frac{1}{2}}\left(\frac{g\alpha a T_0}{\omega}\right)F(\xi, \eta, \tau) \qquad ...(12)$$

and $\sigma = \nu_1/K$ (the Prandtl number), $\nu_2\omega/\nu_1 = -\beta$, where β is positive and of order unity. The appropriate velocity scale is $V = g\alpha a T_0/\omega$. Writing $\sin\theta = S(\xi)$ and $\epsilon = g\alpha a T_0/R\omega^2$, eqns. (4) and (5) become

$$\frac{\partial^2 F}{\partial\eta\partial\tau} - \frac{\sigma}{2}\frac{\partial^3 F}{\partial\eta^3} + \tfrac{1}{2}\beta\sigma\frac{\partial^4 F}{\partial\tau\partial\eta^3} = G\,S(\xi) + \epsilon\left[\frac{\partial F}{\partial\xi}\frac{\partial^2 F}{\partial\eta^2} - \frac{\partial F}{\partial\eta}\frac{\partial^2 F}{\partial\xi\partial\eta}\right]$$
$$- \tfrac{1}{2}\beta\sigma\epsilon\left[\frac{\partial^2 F}{\partial\xi\partial\eta}\frac{\partial^3 F}{\partial\eta^3} - \frac{\partial^2 F}{\partial\eta^2}\frac{\partial^3 F}{\partial\xi\partial\eta^2}\right.$$
$$\left. - \frac{\partial F}{\partial\xi}\frac{\partial^4 F}{\partial\eta^4} + \frac{\partial F}{\partial\eta}\cdot\frac{\partial^4 F}{\partial\xi\partial\eta^3}\right] \qquad ...(13)$$

$$\frac{\partial G}{\partial\tau} - \tfrac{1}{2}\frac{\partial^2 G}{\partial\eta^2} = \epsilon\left[\frac{\partial F}{\partial\xi}\frac{\partial G}{\partial\eta} - \frac{\partial F}{\partial\eta}\frac{\partial G}{\partial\xi}\right]. \qquad ...(14)$$

The boundary conditions (8) become

$$F = \frac{\partial F}{\partial \eta} = 0, \quad G = \cos \omega t, \text{ at } \eta = 0$$

$$\frac{\partial F}{\partial \eta} \to 0, \quad G \to 0 \text{ as } \eta \to \infty \qquad \text{...(15)}$$

We expand $F(\xi, \eta, \tau)$ and $G(\xi, \eta, \tau)$ in powers of ϵ in the following forms :

$$F = S(\xi)\, F_0(\eta)\, e^{i\tau} + \epsilon\, S \frac{dS}{d\xi} \left[F_{1s}(\eta) + F_{1f}(\eta)\, e^{2i\tau}\right] + O(\epsilon^2) \qquad \text{...(16)}$$

$$G = G_0(\eta) e^{i\tau} + \epsilon \frac{dS}{d\xi} \left[G_{1s}(\eta) + G_{1f}(\eta)\, e^{2i\tau}\right] + O(\epsilon^2) \qquad \text{...(17)}$$

where only real parts are to be taken.

The boundary conditions (8) now become

$$\left.\begin{aligned} &F_0 = F_0' = F_{1s} = F_{1s}' = F_{1f} = F_{1f}' = G_{1s} = G_{1f} = 0, \\ &\qquad G_0 = 1, \text{ at } \eta = 0 \\ &F_0' \to 0, F_{1s}' \to 0, F_{1f}' \to 0, G_0 \to 0, G_{1s} \to 0, G_{1f} \to 0 \text{ as } \eta \to \infty \end{aligned}\right\} \text{...(18)}$$

where a prime denotes a differentiation with respect to η. The solution of the resulting equations which are independent of ϵ are,

$$G_0 = \exp\{-(1+i)\,\eta\} \qquad \text{...(19)}$$

$$\begin{aligned} F_0 = &-\frac{A}{B} \exp\left\{\left(-e^{i((\pi/2)-\gamma)}/A\right)\eta - i\left(\varphi + \frac{\pi}{2} - \gamma\right)\right\} \\ &+ \frac{1}{\sqrt{2}\,B} \exp\left\{-(1+i)\,\eta - i\left(\varphi + \frac{\pi}{4}\right)\right\} \\ &+ \frac{A}{B} \exp\left\{-i\left(\frac{\pi}{2} - \gamma + \varphi\right)\right\} - \frac{1}{\sqrt{2}\,B} \exp\left\{-i\left(\frac{\pi}{4} + \varphi\right)\right\} \end{aligned} \qquad \text{...(20)}$$

where

$$A = \left(\frac{\sigma}{2}\right)^{\frac{1}{2}} (1 + \beta^2)^{1/4}, \quad \gamma = \tfrac{1}{2} \tan^{-1}(1/\beta)$$

$$B = \{\sigma^2\beta^2 + (\sigma - 1)^2\}^{1/2}, \quad \tan\phi = (\sigma - 1)/\sigma\beta.$$

The solutions of the equations of order ϵ are

$$G_{1f} = -\frac{\sqrt{2}\,A}{B}\,\Gamma_1 \exp\left\{-(1+i)\,\eta - \left(\frac{1}{A}\,e^{i((\pi/2)-\gamma)}\right)\eta - i\left(\frac{\pi}{4} - \gamma + \varphi + \gamma_1\right)\right\}$$
$$+ \frac{B}{4}\exp\left\{-2\eta - i\left(2\eta + \varphi + \frac{\pi}{2}\right)\right\}$$
$$+ \frac{A}{\sqrt{2}\,B}\exp\left\{-\eta - i\left(\gamma + \eta + \varphi - \frac{\pi}{4}\right)\right\}$$
$$- \frac{1}{2B}\exp\left\{-\eta - i\left(\eta + \varphi - \frac{\pi}{2}\right)\right\} + K_1 \exp\{-\sqrt{2}\,(1+i)\,\eta\} \qquad \ldots(21)$$

$$G_{1s} = -\frac{\sqrt{2}\,A}{B}\,\Gamma_2 \exp\left\{-(1+i)\eta - \left(\frac{1}{A}\,e^{i(\pi/2-\gamma)}\right)\eta - i\left(\frac{3\pi}{4} - \gamma + \varphi + \gamma_2\right)\right\}$$
$$+ \frac{1}{4B}\exp\left\{-2\eta + i\left(\frac{\pi}{2} + \varphi\right)\right\}$$
$$+ \frac{A}{\sqrt{2}\,B}\exp\left\{-\eta + i\left(\eta - \gamma - \frac{\pi}{4} - \varphi\right)\right\}$$
$$- \frac{1}{2B}\exp\{-\eta + i(\eta - \varphi)\} + K_2 \qquad \ldots(22)$$

where

$$\Gamma_1^{-2} = \left\{\frac{2}{A}(\sin\gamma - \cos\gamma) - \frac{1}{A^2}\cos 2\gamma\right\}^2 + 4\left(\frac{1}{A}\sin\gamma + \frac{1}{A}\cos\gamma + \frac{1}{A}\sin\gamma\cos\gamma - 1\right)^2$$

$$\tan\gamma_1 = \frac{\left\{2\left(\frac{1}{A}\sin\gamma + \frac{1}{A}\cos\gamma + \frac{1}{A^2}\sin\gamma\cos\gamma - 1\right)\right\}}{\left\{\frac{2}{A}(\sin\gamma - \cos\gamma) - \frac{1}{A^2}\cos 2\gamma\right\}}$$

$$\Gamma_2^{-2} = \left\{\frac{2}{A}(\sin\gamma + \cos\gamma) - \frac{1}{A^2}\cos 2\gamma\right\}^2 + 4\left(1 + \frac{1}{A}\sin\gamma\right)^2\left(1 - \frac{\cos\gamma}{A}\right)^2$$

$$\tan\gamma_2 = \frac{\{2\,(A + \sin\gamma)(\cos\gamma - A)\}}{\{2\,A\,(\sin\gamma + \cos\gamma) - \cos 2\gamma\}}$$

$$K_1 = \frac{\sqrt{2}\,A}{B}\,\Gamma_1 \exp\left\{-i\left(\frac{\pi}{4}+\gamma+\varphi+\gamma_1\right)\right\}$$
$$-\frac{B}{4}\exp\left\{-i\left(\varphi+\frac{\pi}{2}\right)\right\}$$
$$-\frac{A}{\sqrt{2}\,B}\exp\left\{-i\left(\gamma+\varphi-\frac{\pi}{4}\right)\right\}+\frac{1}{2B}\exp\left\{-i\left(\varphi-\frac{\pi}{2}\right)\right\}$$

$$K_2 = \frac{\sqrt{2}\,A}{B}\,\Gamma_2 \exp\left\{i\left(-\frac{3\pi}{4}+\gamma-\varphi-\gamma_2\right)\right\}$$
$$-\frac{1}{4B}\exp\left\{i\left(\frac{\pi}{2}+\varphi\right)\right\}-\frac{A}{\sqrt{2B}}\exp\left\{-i\left(\gamma+\frac{\pi}{4}+\varphi\right)\right\}$$
$$+\frac{1}{2B}\exp(-i\varphi).$$

Using (16) to (22) in the equation (13) we get the following equations for F_{1f} and F_{1s} :

$$\left(\beta\sigma i-\frac{\sigma}{2}\right)F_{1f}''' + 2iF_{1f}' = \chi_{1f}(\eta), \qquad \text{...(23)}$$

$$F_{1s}''' = \chi_{1s}(\eta), \qquad \text{...(24)}$$

where χ_{1f} and χ_{1s} are functions of η. We have obtained the expressions for F_{1f} and F_{1s} using additional boundary conditions $F_{1f}'' \to 0$, $F_{1s}'' \to 0$ as $\eta \to \infty$. These expressions are very lengthy and hence they have not been reproduced here.

4. Discussion

It is seen that all the expressions giving velocity and temperature fields depend on the non-Newtonian parameter β. The expressions for G_{1s} and F_{1s} do not satisfy boundary conditions at large distances from the cylinder, but these will give correct information for the flow near the cylinder specially for calculating the rate of heat transfer from the cylinder and the force experienced by it.

The heat flux per unit area and time at a point θ is given by

$$q_\theta = -k\left[\frac{\partial T}{\partial y}\right]_{y=0} = -kaT_0(\omega/2K)^{1/2}\left[\frac{\partial G}{\partial \eta}\right]_{\eta=0}$$

where k is the thermal conductivity. The non-dimensional rate of heat flux per unit area and time is given by (neglecting terms of $O(\epsilon^2)$ and higher order of ϵ)

$$-\left[\frac{\partial G}{\partial \eta}\right]_{\eta=0} = -\left[G_0'\,e^{i\tau}+\epsilon\frac{dS}{d\xi}\left\{G_{1s}'+G_{1f}'\,e^{2i\tau}\right\}\right]_{\eta=0}.$$

The values of $\left[G'_{1s}\right]_{\eta=0}$ have been tabulated in a bivariate Table I for various values of β and σ. If we want to find out the value of $\left[G'_{1s}\right]_{\eta=0}$, say, for $\beta = 0.4$ and $\sigma = 20$, it can be obtained by taking the intersection of the column containing $\beta = 0.4$ and the row containing $\sigma = 20$ which reads as 0.1170. It is seen that the purely oscillatory temperature distribution of the cylinder induces a steady heat transfer rate from the cylinder. This steady heat transfer rate is negative, i.e., the cylinder absorbs heat as long as $dS/d\xi$ is positive and when $dS/d\xi$ is negative heat is being transferred from the cylinder to the fluid. For example, in the case of a circular cylinder there is a steady state heat transfer rate from the fluid to the lower half of the cylinder and this situation is reversed for the upper half of the cylinder. The magnitude of this steady state heat transfer rate increases with the increase of β as well as with the decrease of the Prandtl number σ.

TABLE I

Bivariate table of $\left[G'_{1s}\right]_{\eta=0}$ *for various values of* β *and* σ.

σ \ β	0	0.3	0.4	0.5
10	0.1321	0.2044	0.2093	0.2104
20	0.0595	0.1102	0.1170	0.1195
40	0.0279	0.0640	0.0697	0.0726
60	0.0182	0.0478	0.0527	0.0554
80	0.0135	0.0391	0.0436	0.0461
100	0.0107	0.0337	0.0377	0.0401
150	0.0070	0.0259	0.0293	0.0313

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