EFFECT OF CAPILLARY FORCE ON KELLER'S SOLITARY WAVE

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The main object of this paper is to include the capillary force at the free surface along with the vertical force of gravity. The boundary condition of the continuity of pressure at the free surface is replaced by the discontinuity condition denoted by $p - p_0 = T/R$, where p denotes the pressure at any point of the free surface $y = \eta(x, t)$ from the liquid side and p_0 the constant atmospheric pressure on the top of it, T the surface tension and R the radius of curvature at the free surface. The results obtained are of the same form as those obtained in the Keller's theory for purely gravitating solitary wave, however, they differ with them in having $x(1 + 3\gamma)^{-1/2}$ instead of x where $\gamma = T/g\rho h^2$. It is found that the peak of the solitary wave is widened on account of the surface tension.

INTRODUCTION

Upto the present time a fairly extensive literature is available on the solitary wave of Scott Russell and cnoidal wave of Korteweg and de Vries. A systematic treatment of these waves come under the scope of non-linear shallow-water theory derived by Friedrichs. In fact, Keller (1948), using the systematic perturbation method of Friedrichs has shown that in steady motion the second order approximation gives cnoidal wave which in the limiting case when the wavelength approach infinity becomes a solitary wave.

In the above problem liquid is heavy and homogeneous and the motion is two-dimensional, irrotational. The main object of this paper is to include the capillary force at the free surface along with the vertical force of gravity in the Keller's theory and obtain the modified cnoidal and solitary waves. In other words, the dynamical boundary condition of the continuity of pressure at the free surface is replaced by the discontinuity condition mentioned by the surface tension and the problem is solved exactly on the lines of Keller. It is found that the peak of the solitary wave is widened on account of the surface tension.

FORMULATION

A two-dimensional irrotational motion of homogeneous inviscid liquid is considered with reference to a frame of reference XOY whose Y-axis is vertically upwards. The liquid is contained in a straight infinite channel extending from $x = -\infty$ to $x = +\infty$, where the bed as given by the equation $y = d(x)$ is rigid. In addition to the vertical force of gravity with corresponding acceleration denoted by g, a constant surface tension is considered at the free surface of the liquid which maintains a difference of pressure from the two sides of the free surface.

If p denotes the dynamical pressure at any point of the free surface $y = \eta(x, t)$ from the liquid side and p_0 the constant atmospheric pressure on the top of it, we shall have $p - p_0 = T/R$, where T denotes the constant surface tension and R the radius of curvature of the free surface, counted positive when the centre of curvature lies on the upper side of the free surface, i.e., when d^2y/dx^2 is positive. This dynamical type of boundary condition is satisfied on the free surface whose form is not known a priori. This inherent difficulty of unknown domain of flow can be overcome in the case of a potential two-dimensional flow if the velocity distribution and the surface elevation $y = \eta(x, t)$ are determined in terms of velocity potential φ and the stream function ψ. In fact, the theory of Friedrichs and Hyers (1954) for the existence of a purely gravitational solitary wave begins with such a formulation of the problem in the complex potential plane $f(z) = \varphi(x, y) + i\,\psi(x, y)$ on the lines devised by Levi-Civita (1925) for treating the existence of periodic waves of finite amplitude. The present problem of finding a capillary affected solitary wave can also be treated on the lines of Friedrichs and Hyers theory. Here, in this article the problem is treated on the lines of Keller's theory and the modified cnoidal wave is first obtained which in the limiting case of the wavelength tending to infinity becomes a capillary affected solitary wave.

The equations of motion and the boundary conditions for the determination of the velocity components $\bar{u}(x, y, t)$, $\bar{v}(x, y, t)$ pressure $p(x, y, t)$ and the surface elevation $\eta(x, t)$ are given by

$$\left.\begin{aligned}
\bar{u}_x + \bar{v}_y &= 0 & &\\
\bar{u}_t + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y &= -\,p_x/\rho & &\\
\bar{v}_t + \bar{u}\bar{v}_x + \bar{v}\bar{v}_y &= -\,(p_y/\rho) - g & &\\
\bar{v}_x &= \bar{u}_y & &\\
\eta_t + \bar{u}\eta_x &= \bar{v} & &\text{at } y = \eta(x, t)\\
p - p_0 &= T\,\eta_{xx}\,\{1 + (\eta_x)^2\}^{-3/2} & &\text{at } y = \eta(x, t)\\
\bar{u}d_x &= \bar{v} & &\text{at } y = d(x).
\end{aligned}\right\} \qquad \ldots(1)$$

As in the Friedrichs's systematic shallow water theory the dimensional variables above are changed to their corresponding non-dimensional form by the transformations :

$$\left.\begin{array}{lll} x = \alpha\omega^{-1}, & \tau = (gh)^{1/2}\,\omega t, & \bar{u} = (gh)^{1/2}\,u \\ y = \beta h, & \eta = Yh, & \bar{v} = (gh)^{1/2}\,(\omega h)^{-1}\,v \\ d = Hh, & p - p_0 = g\rho h\pi, & \sigma = \omega^2 h^2, \quad \gamma = T/g\rho h^2 \end{array}\right\} \quad \ldots(2)$$

where the constants h and ω^{-1} denote the typical length in the vertical and horizontal directions respectively. σ and γ are non-dimensional parameters.

In terms of the new non-dimensional variables the above equations and boundary conditions in (1) become

$$\left.\begin{array}{ll} \sigma\, u_\alpha + v_\beta = 0 & \\ \sigma\,[u_\tau + uu_\alpha + \pi_\alpha] + vu_\beta = 0 & \\ \sigma\,[v_\tau + uv_\alpha + 1 + \pi_\beta] + vv_\beta = 0 & \\ v_\alpha = u_\beta & \\ \sigma\,[Y_\tau + uY_\alpha] = v & \text{at } \beta = Y \\ \pi = \gamma\sigma Y_{\alpha\alpha}\,[1 + \sigma(Y_\alpha)^2]^{-3/2} & \text{at } \beta = Y \\ \sigma u H_\alpha = v & \text{at } \beta = H. \end{array}\right\} \quad \ldots(3)$$

It is to be noted that mathematically this problem differs from the Keller's problem in having $\pi = \gamma\sigma Y_{\alpha\alpha}\,[1 + \sigma\,(Y_\alpha)^2]^{-3/2}$ at $\beta = Y$ instead of $\pi = 0$ at $\beta = Y$ in the latter case.

To solve this problem the following power series expansions for u, v, π and Y are assumed :

$$\left.\begin{array}{l} u = u^{(0)} + \sigma u^{(1)} + \sigma^2 u^{(2)} + \ldots \\ v = v^{(0)} + \sigma v^{(1)} + \sigma^2 v^{(2)} + \ldots \\ \pi = \pi^{(0)} + \sigma\pi^{(1)} + \sigma^2\pi^{(2)} + \ldots \\ Y = Y^{(0)} + \sigma Y^{(1)} + \sigma^2 Y^{(2)} + \ldots \end{array}\right\} \quad \ldots(4)$$

where each $u^{(i)}$, $v^{(i)}$, $\pi^{(i)}$ is a function of α, β and τ and $Y^{(i)}$ are functions of α and τ only.

In order to obtain cnoidal waves we shall be concerned with the steady state solution only. Following the method of Keller's theory we insert the above series in the system (3) and get from the coefficients of σ^0, σ^1, σ^2 and σ^3 the following sets of equations and boundary conditions for u^i, v^i, π^i and Y^i ($i = 0, 1, 2, 3$) :

$$\left.\begin{array}{l} v^0_\beta = 0 \\ v^0 u^0_\beta = 0 \end{array}\right\} \quad \ldots(5a)$$

$$\left.\begin{aligned}
&v^0 v^0_\beta = 0 \\
&v^0_\alpha = u^0_\beta \\
&v^0 = 0 && \text{at } \beta = Y^0 \\
&\pi^0 = 0 && \text{at } \beta = Y^0 \\
&v^0 = 0 && \text{at } \beta = H.
\end{aligned}\right\} \qquad \ldots(5b)$$

$$\left.\begin{aligned}
&u^0_\alpha + v^1_\beta = 0 \\
&u^0 u^0_\alpha + \pi^0_\alpha + v^0 u^1_\beta + v^1 u^0_\beta = 0 \\
&u^0 v^0_\alpha + 1 + \pi^0_\beta + v^0 v^1_\beta + v^1 v^0_\beta = 0 \\
&v^1_\alpha = u^1_\beta \\
&u^0 Y^0_\alpha = v^1 + Y^1 v^0_\beta && \text{at } \beta = Y^0 \\
&\pi^1 + Y^1 \pi^0_\beta = \gamma Y^0_{\alpha\alpha} && \text{at } \beta = Y^0 \\
&u^0 H_\alpha = v^1 && \text{at } \beta = H.
\end{aligned}\right\} \qquad \ldots(6)$$

$$\left.\begin{aligned}
&u^1_\alpha + v^2_\beta = 0 \\
&u^0 u^1_\alpha + u^0_\alpha u^1 + \pi^1_\alpha + v^0 u^2_\beta + v^1 u^1_\beta + v^2 u^0_\beta = 0 \\
&u^0 v^1_\alpha + u^1 v^0_\alpha + \pi^1_\beta + v^0 v^2_\beta + v^1 v^1_\beta + v^2 v^0_\beta = 0 \\
&v^2_\alpha = u^2_\beta \\
&u^0 Y^1_\alpha + u^1 Y^0_\alpha = v^2 + Y^1 v^1_\beta + Y^2 v^0_\beta && \text{at } \beta = Y^0 \\
&\pi^2 + Y^1 \pi^1_\beta + Y^2 \pi^0_\beta = \gamma Y^1_{\alpha\alpha} && \text{at } \beta = Y^0 \\
&u^1 H_\alpha = v^2 && \text{at } \beta = H.
\end{aligned}\right\} \qquad \ldots(7)$$

$$\left.\begin{aligned}
&u^2_\alpha + v^3_\beta = 0\\
&u^0u^2_\alpha + u^1u^1_\alpha + u^2u^0_\alpha + \pi^2_\alpha + v^0u^3_\beta + v^1u^2_\beta + v^2u^1_\beta + v^3u^0_\beta = 0\\
&u^0v^2_\alpha + u^1v^1_\alpha + u^2v^0_\alpha + \pi^2_\beta + v^0v^3_\beta + v^1v^2_\beta + v^2v^1_\beta + v^3v^0_\beta = 0\\
&v^3_\alpha = u^3_\beta\\
&u^0Y^2_\alpha + u^1Y^1_\alpha + u^2Y^0_\alpha = v^3 + Y^1v^2_\beta + Y^2v^1_\beta + Y^3v^0_\beta \quad \text{at } \beta = Y^0\\
&\pi^3 + Y^1\pi^2_\beta + Y^2\pi^1_\beta + Y^3\pi^0_\beta = \gamma Y^2_{\alpha\alpha} \quad \text{at } \beta = Y^0\\
&H_\alpha\left\{u^2 + Y^2u^0_\beta + Y^1u^1_\beta\right\} = v^3 \quad \text{at } \beta = H.
\end{aligned}\right\} \quad \ldots(8)$$

From the set (5a, b) we get

$$v^0(\alpha, \beta) \equiv 0,\ u^0 = u^0(\alpha),\ \pi^0(\alpha, Y^0) = 0. \quad \ldots(9)$$

From the first and last of the equations in (6) we obtain

$$v^1 = -\beta u^0_\alpha + (u^0H)_\alpha. \quad \ldots(10)$$

Third equation of the set (6) gives the hydrostatic pressure relation

$$\pi^0 = Y^0 - \beta. \quad \ldots(11)$$

From the sixth of the equations in (6) we obtain

$$\pi^1 = Y^1 + \gamma Y^0_{\alpha\alpha} \quad \text{at } \beta = Y^0 \quad \ldots(12)$$

and the fourth equation of the same set gives

$$u^1 = -\frac{\beta^2}{2}u^0_{\alpha\alpha} + \beta\,(u^0H)_{\alpha\alpha} + f(\alpha) \quad \ldots(13)$$

where $f(\alpha)$ is an arbitrary function.

Using (10) and (11) the second and the fifth equations in (6) separately give

$$\left.\begin{aligned}
&u^0u^0_\alpha + Y^0_\alpha = 0\\
\text{and}\quad &[u^0(Y^0 - H)]_\alpha = 0
\end{aligned}\right\} \quad \ldots(14)$$

which on integration become

$$\tfrac{1}{2}(u^0)^2 + Y^0 = e$$

and $$u^0(Y^0 - H) = m$$

where e and m are constants. Eliminating u^0 we get the cubic equation in $(Y^0 - H)$

$$(Y^0 - H)^3 - (e - H)(Y^0 - H)^2 + \frac{m^2}{2} = 0.$$

If we assume that the bottom is horizontal the only continuous steady state will be given by Y^0 = constant and u^0 = constant.

In the case of horizontal bottom, since Y^0 = const., u^0 = const., we get from (10), (12) and (13) respectively the following results :

$$\left.\begin{aligned} v^1 &= 0 \\ \pi^1 &= Y^1 \qquad \text{at } \beta = Y^0 \\ u^1 &= f(\alpha). \end{aligned}\right\} \qquad \ldots(15)$$

Keeping in mind the results obtained in (9) and (15) the first and last of the equations of the set (7) will give

$$v^2 = -(\beta - H)u^1_\alpha . \qquad \ldots(16)$$

Third of the equations of the set (7) gives

$$\pi^1 = g(\alpha) \qquad \ldots(17)$$

where $g(\alpha)$ is an arbitrary function. Combining this with (15) we get

$$\pi^1 = Y^1 = g(\alpha). \qquad \text{at } \beta = Y^0. \qquad \ldots(18)$$

From the fourth of the equations (7) we obtain

$$u^2 = -\frac{(\beta - H)^2}{2} u^1_{\alpha\alpha} + r(\alpha) \qquad \ldots(19)$$

where $r(\alpha)$ is an arbitrary function.

The second and fifth of the equations in (7) separately give

$$\left.\begin{aligned} &u^0 u^1_\alpha + Y^1_\alpha = 0 \\ \text{and} \quad & \\ &u^0 Y^1_\alpha + (Y^0 - H)u^1_\alpha = 0. \end{aligned}\right\} \qquad \ldots(20)$$

The only solution of these homogeneous linear equations is $u^1_\alpha = Y^1_\alpha = 0$ and therefore u^1 = constant and Y^1 = constant, unless the determinant is zero, i.e. if $(u^0)^2 = Y^0 - H$

and in this case $Y^1_\alpha = -u^0 u^1_\alpha$, where u^1_α is arbitrary. This means that $\bar{u}^0=(g\,.\text{depth})^{1/2}$, which is the critical speed or the propagation speed in the shallow water theory. Therefore, at the critical speed $(u^0)^2 = Y^0 - H$ we get

$$Y^1 = -u^0 u^1 + c \qquad ...(21)$$

where c is a constant.

The sixth of the equations in (7) gives

$$\pi^2 = Y^2 + \gamma Y^1_{\alpha\alpha} \qquad \text{at } \beta = Y^0. \qquad ...(22)$$

We now proceed to solve the set of the equations in (8) after substituting the values of v^2, v^1, v^0 and u^2 already determined from the sets (5), (6) and (7). Third of the equations gives

$$\pi^2 = \frac{(\beta - H)^2}{2} u^0 u^1_{\alpha\alpha} + s(\alpha) \qquad ...(23)$$

where $s(\alpha)$ is an arbitrary function, so that from (22) we get

$$Y^2 = \frac{(Y^0 - H)^2}{2} u^0 u^1_{\alpha\alpha} + s(\alpha) - \gamma Y^1_{\alpha\alpha}. \qquad ...(24)$$

From the first and last equations in (8) we shall have

$$v^3 = \frac{(\beta - H)^3}{6} u^1_{\alpha\alpha\alpha} - (\beta - H)\, r_\alpha. \qquad ...(25)$$

Using the value of π^2 and u^2 in the second equation of the set (8) we get

$$s_\alpha + u^0 r_\alpha = -\tfrac{1}{2}(u^1)^2_\alpha$$

which on integration gives

$$s + u^0 r = -\tfrac{1}{2}(u^1)^2 + b \qquad ...(26)$$

where b is a constant.

Fifth of the equations in (8) becomes

$$u^0 Y^2_\alpha + (u^1 Y^1)_\alpha = \frac{(Y^0 - H)^3}{6} u^1_{\alpha\alpha\alpha} - (Y^0 - H)\, r_\alpha$$

which on integration yields

$$u^0 Y^2 + u^1 Y^1 = \frac{(Y^0 - H)^3}{6} u^1_{\alpha\alpha} - (Y^0 - H)\, r + j \qquad ...(27)$$

where j is a constant.

From (24) and (26) we get

$$Y^2 = \frac{(Y^0 - H)^2}{2} u^0 u^1_{\alpha\alpha} - \gamma Y^1_{\alpha\alpha} - u^0 r - \tfrac{1}{2}(u^1)^2 + b.$$

Using this value of Y^2 in (27) we get

$$\frac{(Y^0 - H)^2}{2}(u^0)^2 u^1_{\alpha\alpha} - \gamma u^0 Y^1_{\alpha\alpha} - (u^0)^2 r - \tfrac{1}{2}u^0(u^1)^2 + u^0 b + Y^1 u^1$$
$$= -(Y^0 - H)r + \frac{(Y^0 - H)^3}{6} u^1_{\alpha\alpha} + j$$

and since

$$(u^0)^2 = (Y^0 - H) \text{ and } Y^1 = -u^0 u^1 + c$$

we get

$$\left\{\gamma (u^0)^2 + \frac{(u^0)^6}{3}\right\} u^1_{\alpha\alpha} = \frac{3}{2} u^0 (u^1)^2 - c u^1 + j - b u^0$$

which on integration gives

$$(u^1_\alpha)^2 = \frac{1}{\{\gamma u^0 + \frac{1}{3}(u^0)^5\}} \Big[(u^1)^3 - \frac{c}{u^0}(u^1)^2 + \frac{2}{u^0}(j - b u^0) u^1 + q\{\gamma u^0 + \tfrac{1}{3}(u^0)^5\} \Big] \qquad \ldots(28)$$

where q is a constant. This equation can be integrated in terms of the elliptic function.

In order to reduce this into the standard form, in the case when all the zeros of the cubic in u^1 in the right-hand big bracket has real roots denoted by l, h_1, h_2 $(l \geqslant h_1 \geqslant h_2)$ we introduce a variable χ by the relation

$$u^1 = h_1 - (h_1 - h_2)\cos^2 \chi \qquad \ldots(29)$$

where

$$\triangle^{-1} = \tfrac{1}{2}\left\{\frac{l - h_2}{\gamma u^0 + \frac{1}{3}(u^0)^5}\right\}^{1/2} \text{ and } K^2 = \frac{h_1 - h_2}{l - h_2}.$$

Evidently the roots l, h_1 and h_2 will in general depend on γ.

Equation (28) then reduces to the standard form

$$\frac{d\chi}{d\alpha} = \triangle^{-1}(1 - K^2 \sin^2 \chi)^{1/2}. \qquad \ldots(30)$$

Integrating this we get

$$\alpha - \alpha_0 = \triangle \int_{\chi_0}^{\chi} (1 - K^2 \sin^2 \chi)^{-1/2} \, d\chi.$$

We choose the origin in such a way that $\alpha_0 = \chi_0 = 0$ which means that the origin occurs where $u^1 = h_2$. Thus

$$\alpha = \triangle \int_0^{\chi} (1 - K^2 \sin^2 \chi)^{-1/2}\, d\chi = \triangle\, F(\chi, K)$$

where $F(\chi, K)$ is the elliptic integral of the first kind and of modulus K.

The inverse function is

$$\chi = \text{arc}\ \cos cn\ \frac{\alpha}{\triangle} (\text{mod}\ K)$$

and thus from (29) we get

$$u^1 = h_1 - (h_1 - h_2)\, cn^2\ \frac{\alpha}{\triangle} (\text{mod}\ K) \qquad \ldots(31)$$

which is periodic in α.

The wave length is then

$$\lambda = 2 \triangle \int_0^{\pi/2} (1 - K^2 \sin^2 \chi)^{-1/2}\, d\chi = 2 \triangle\, F(\pi/2, K) = 2 \triangle\, F_1(K). \qquad \ldots(32)$$

And from (21) we obtain

$$\pi^1 = Y^1 = - u^0 u^1 + c = c - u^0 h_1 + u^0 (h_1 - h_2)\, cn^2\ \frac{\alpha}{\triangle} (\text{mod}\ K). \qquad \ldots(33)$$

Further, if Y be elevation above the mean level we have

$$\int_0^{\lambda} Y d\alpha = \lambda Y^0$$

which means that

$$\int_0^{\lambda} Y^1 d\alpha = 0$$

in the second approximation.

This gives

$$c = u^0 l - u^0 (l - h_2) \frac{E_1}{F_1}$$

and as $l + h_1 + h_2 = \frac{c}{u^0}$ we have

$$(h_2 - l)\, E_1(K) = (h_1 + h_2)\, F_1(K) \qquad \ldots(34)$$

where $E_1(K)$ is the elliptic integral of the second kind.

Finally, combining the first and second approximations, we get

$$\left.\begin{aligned} v &= v^0 + \sigma v^1 = 0 \\ u &= u^0 + \sigma u^1 = u^0 + \sigma h_1 - (\sigma h_1 - \sigma h_2)\, cn^2 \frac{\alpha}{\Delta} (\text{mod } K) \\ Y &= Y^0 + \sigma Y^1 = Y^0 + u^0(\sigma l + \sigma h_2) \\ &\qquad + u^0(\sigma h_1 - \sigma h_2)\, cn^2 \frac{\alpha}{\Delta} (\text{mod } K) \\ \pi &= \pi^0 + \sigma \pi^1 = Y^0 - \beta + u^0(\sigma l + \sigma h_2) \\ &\qquad + u^0(\sigma h_1 - \sigma h_2)\, cn^2 \frac{\alpha}{\Delta} (\text{mod } K). \end{aligned}\right\} \quad \ldots(35)$$

Since $cn^2 \frac{\alpha}{\Delta}$ varies between zero and one the maximum and minimum values of Y are given by

$$Y_{max} = Y^0 + u^0(\sigma l + \sigma h_1)$$
$$Y_{min} = Y^0 + u^0(\sigma l + \sigma h_2)$$

Assuming $H = 0$, which means that a vertical distance is measured from the bottom and that h is the average height of the surface, the equations (35), (34) and (32) can be rewritten in terms of the original coordinates as follows (writing L for $\sigma\, l$):

$$\bar{v} = 0$$

$$\frac{\bar{u}}{(gh)^{1/2}} = \frac{\eta_{max}}{h} - L - \left(\frac{\eta_{max}}{h} - \frac{\eta_{min}}{h}\right) cn^2 \left[\frac{x}{\lambda}\, 2F_1(K)\right] (\text{mod } K)$$

$$\eta = \eta_{min} + (\eta_{max} - \eta_{min}) cn^2 \left[\frac{x}{\lambda}\, 2F_1(K)\right] (\text{mod } K)$$

$$\frac{p - p_0}{g\rho} = \eta - y$$

$$\frac{\bar{\lambda}}{h} = \frac{4}{3^{1/2}} (1 + 3\gamma)^{1/2} F_1(K) \left(2L + 1 - \frac{\eta_{min}}{h}\right)^{-1/2}$$

where

$$2L + 1 > \frac{\eta_{max}}{h} > \frac{\eta_{min}}{h},\; 0 < K^2 = \frac{\dfrac{\eta_{max}}{h} - \dfrac{\eta_{min}}{h}}{2L + 1 - \dfrac{\eta_{min}}{h}} \leqslant 1$$

and

$$\left(2L + 1 - \frac{\eta_{min}}{h}\right) E_1(K) = \left(2L + 2 - \frac{\eta_{max}}{h} - \frac{\eta_{min}}{h}\right) F_1(K).$$

Equations (36) represent the steady state solutions. In the extreme case when $\bar{\lambda} = \infty$ these equations yield the solitary wave given by

$$\bar{v} = 0, \bar{\lambda} = \infty,$$

$$\frac{\bar{u}}{(gh)^{1/2}} = \tfrac{1}{2}\left(\frac{\eta_{max}}{h} + 1\right) - \left(\frac{\eta_{max}}{h} - 1\right) \sec h^2 \left\{\frac{3^{1/2}\, x}{2h} (1 + 3\gamma)^{-1/2} \times \left(\frac{\eta_{max}}{h} - 1\right)^{1/2}\right\}$$

$$\frac{\eta}{h} = 1 + \left(\frac{\eta_{max}}{h} - 1\right) \sec h^2 \left\{\frac{3^{1/2}\, x}{2h} (1 + 3\gamma)^{-1/2} \left(\frac{\eta_{max}}{h} - 1\right)^{1/2}\right\}$$

$$\frac{p - p_0}{g\rho} = \eta - y.$$

The above results have the same form as those obtained in Keller's (1948) theory for purely gravitating solitary wave. However, they differ from them in having $x(1 + 3\gamma)^{-1/2}$ instead of x.

As η_{max} will depend on surface tension, the height of the solitary wave will be affected by surface tension, and the peak will generally be widened.

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