INVERSION FORMULAS FOR THE INTEGRAL TRANSFORMATION WITH THE *H*-FUNCTION AS KERNEL

by R. G. BUSCHMAN* *and* H. M. SRIVASTAVA**, *Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada V8W 2Y2*

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For an integral transformation, which is a Mellin convolution with Fox's *H*-function, relations between some of the known forms of inversion are discussed, the difficulties in the inversion given by Singh (1970) are explained, and certain new forms for the inversion are obtained.

§1. We begin by recalling the definition of Fox's *H*-function (1961, p. 408) in the (slightly modified) form

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + \beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j - \alpha_j s)} z^s\, ds \qquad \ldots(1)$$

where an empty product is interpreted as 1, m, n, p, q are integers such that $0 \leqq m \leqq q$, $0 \leqq n \leqq p$, the α_j and β_j are all positive, the parameters are such that no poles of the integrand coincide, and the contour $\mathrm{Re}\,(s) = \sigma$ separates the poles of one product from those of the other. If we let

$$K = \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=n+1}^{p}\alpha_j - \sum_{j=m+1}^{q}\beta_j \qquad \ldots(2)$$

then, by the results of Dixon and Ferrar (1936), the integral in (1) converges for $K > 0$ provided $|\arg(z)| < \frac{1}{2}K\pi$, diverges for $K < 0$, and for $K = 0$ it would converge under certain additional assumptions. For convenience, we shall abbreviate the first member of eqn. (1) by $H_{p,q}^{m,n}[z]$.

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We consider the integral transformation in the format

$$\int_0^\infty x^{-1} H_{p,q}^{m,n}[xy] f(x)\, dx = \hat{f}(y) \qquad \ldots(3)$$

which has some slight notational advantages. This differs from the form used by Gupta and Mittal (1970). In that there a y appears instead of x^{-1} as a factor, but the identity

$$z H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1 + \alpha_1, \alpha_1), \ldots, (a_p + \alpha_p, \alpha_p) \\ (b_1 + \beta_1, \beta_1), \ldots, (b_q + \beta_q, \beta_q) \end{matrix} \right] \qquad \ldots(4)$$

allows conversion from one form to the other. Singh (1970) chooses a similar form and introduces two parameters which seem to be unnecessary.

For cases in which $K = 0$ we note that inversion formulae have been obtained by Fox (1961), Kesarwani (1965) and Saxena (1966; 1967a, b). For $K > 0$, the following inversion theorems have been given; we restate them here to fit our common notation.

Theorem A (Gupta and Mittal 1970) — If $x^{k-1}f(x) \in L(0, \infty)$, and $f(x)$ is of bounded variation in the neighbourhood of the point x, then the integral transformation in (3) has for its inversion

$$\tfrac{1}{2}\{f(x+0) + f(x-0)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\tau-iT}^{\tau+iT} \frac{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)}{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)} x^{-s} F(s)\, ds \qquad \ldots(5)$$

where

$$F(s) = \int_0^\infty u^{s-1} \hat{f}(u)\, du \qquad \ldots(6)$$

provided that $K > 0$, $|\arg(y)| < \frac{1}{2} K\pi$,

$$\max_{1 \leqq j \leqq n} \{\mathrm{Re}\,[(a_j - 1)/\alpha_j]\} < \mathrm{Re}\,(s) = k < \min_{1 \leqq j \leqq m} \{\mathrm{Re}\,(b_j/\beta_j)\} \qquad \ldots(7)$$

and the H-function transform of $|f(t)|$ exists.

Theorem B (Singh 1970) — If $x^{k-1}f(x)$ and $y^{-k-1}\hat{f}(y)$ belong to $L(0, \infty)$, and if $\hat{f}(y)$ is given by (3), then with

$$\left.\begin{aligned} c_j &= 1 - a_{p-j+1},\ \lambda_j = \alpha_{p-j+1},\ 1 \leqq j \leqq p \\ d_j &= 1 - b_{q-j+1},\ \mu_j = \beta_{q-j+1},\ 1 \leqq j \leqq q \end{aligned}\right\} \qquad \ldots(8)$$

$$\hat{f}(x) = \int_0^\infty y^{-1} f(y)\, H_{p,q}^{q-m,p-n}\left[xy \,\middle|\, \begin{matrix} (c_1, \lambda_1), \ldots, (c_p, \lambda_p) \\ (d_1, \mu_1), \ldots, (d_q, \mu_q) \end{matrix} \right] dy \qquad \ldots(9)$$

provided

(i) $0 \leqq n \leqq p$, $0 \leqq m \leqq q$, $\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j < 0$;

(ii) $f(x)$ is continuous at x, $s = k + it$, $0 < t < \infty$;

(iii) $a + \sigma_2 > 0$, $b + \sigma_1 < 0$, where

$$f(x) = \begin{cases} O(|x|^{a}) & \text{for small } x, \\ O(|x|^{b}) & \text{for large } x, \end{cases}$$

and σ_1, σ_2 denote, respectively, the first and last members of (7), and $K > 0$;

(iv) $\sigma_1 < k < \sigma_2$;

(v) the H-function in the inversion integral and the inversion integral exist.

A special case of (3) with $H_{m+n,m+k}^{m,n}$ type of kernel and with $K > 0$ was considered by Saxena (1966) in which the inversion involved fractional integrals.

Our development of the inversion is along the common lines of the three aforementioned inversions. Our object, therefore, is to show how those forms are related, to clarify and correct the results of Singh (1970), and to obtain some additional forms in which the inversion can be presented.

§2. Since our form of the integral eqn. (3) can be conveniently interpreted as a simple Mellin convolution [Erdélyi *et al.* 1954, p. 308, eqn. (13); see also Titchmarsh 1948], and since the Mellin transform of the H-function is given by

$$\mathcal{M}\left\{H_{p,q}^{m,n}[x]; s\right\} = 1/Q(-s), \quad -\sigma_2 < \mathrm{Re}\,(s) < -\sigma_1 \qquad \ldots(10)$$

where $Q(s)$ denotes the quotient of Γ-functions occurring on the right-hand side of (5), we can apply this transform with $-s$ in place of s to our integral equation (3) to obtain

$$[1/Q(s)]\, \mathcal{M}\{f(x); s\} = \mathcal{M}\{\hat{f}(y); -s\}. \qquad \ldots(11)$$

Consequently, if we solve for $\mathcal{M}\{f(x); s\}$ and apply the inverse Mellin transform, we obtain the format

$$\tfrac{1}{2}[f(x+0) + f(x-0)] = \mathcal{M}^{-1}\left\{Q(s)\, \mathcal{M}\{\hat{f}(y); -s\}\right\} \qquad \ldots(12)$$

which is the result of Theorem A.

On the other hand, if we try to use the $Q(s)$ in (12) as the quotient of Γ-functions in the defining relation (1) for the H-function, we find, for the K' associated with this quotient, that $K' = -K < 0$, and the integral diverges. As a result of this we cannot simply invert the product

$$Q(s)\,\mathcal{M}\{\hat{f}(y); -s\} \qquad \ldots(13)$$

as a Mellin convolution and thus the first part of hypothesis (v) of Theorem B can never be satisfied. In this sense, the inversion theorem of Singh (1970) is not useful, except as a formality as was done in his Example III (p. 61) where the Γ-functions were immediately reintroduced in order to produce an inversion of the form of Theorem A. It should also be noted that in the same paper Example I, p. 60, and Example II, p. 61, actually satisfy the condition $K = 0$, not $K > 0$, and hence are not particular cases of Theorem B.

§3. There are various ways in which an H-function can be introduced into the inversion, but only at the expense of complication of the other parts of the inversion; we shall next consider some of them.

One obvious rearrangement to try is to separate a sufficient number of the factors of the denominator of $Q(s)$ so that $Q(s) = Q_1(s)/P(s)$, where the remaining quotient $Q_1(s)$ has an associated constant $K_1 > 0$. Then the inversion can be reorganized into the format of a Mellin convolution

$$f(x) = \mathcal{M}^{-1}\{Q_1(s)\} * \mathcal{M}^{-1}\{\mathcal{M}\{\hat{f}(y); -s\}/P(s)\}. \qquad \ldots(14)$$

If we have

$$P(s) = \prod_{j=1}^{h} \Gamma(b_j - \beta_j s) \prod_{j=1}^{l} \Gamma(1 - a_j + \alpha_j s),\ 0 \leqq h \leqq m,\ 0 \leqq l \leqq n \qquad \ldots(15)$$

and

$$\mathcal{M}\{\hat{f}_1(y); -s\} = [1/P(s)]\,\mathcal{M}\{\hat{f}(y); -s\} \qquad \ldots(16)$$

then, at a point of continuity of $f(x)$,

$$f(x) = \int_0^\infty y^{-1} H_{p-l,q-h}^{q-m,p-n}\left[xy \,\middle|\, \begin{matrix} (c_1, \lambda_1), \ldots, (c_{p-l}, \lambda_{p-l}) \\ (d_1, \mu_1), \ldots, (d_{q-h}, \mu_{q-h}) \end{matrix} \right] \hat{f}_1(y)\, dy \qquad \ldots(17)$$

where the parameters of the H-function are given by (8).

Alternatively, it may be useful to multiply $Q(s)$ by an appropriate factor involving Γ-functions in order to form a new quotient $Q_2(s)$ with $K_2 > 0$. Cases which come immediately to mind are factors $\Gamma(1 - c + \gamma s)$, $\Gamma(c - \gamma s)$, $\{\Gamma(1 - c + s)\}^N$ or $\{\Gamma(c - s)\}^N$, or some other combination of these. For the first two cases γ is to be

so chosen that $K_2 = \gamma - K > 0$, and for the next two cases, $K_2 = N - K > 0$. For example, if

$$Q_2(s) = Q(s)\,\Gamma(1 - c + \gamma s) \qquad \ldots(18)$$

then we have

$$f(x) = \int_0^\infty y^{-1} H_{p,q+1}^{q-m+1,p-n}\left[xy \,\middle|\, \begin{matrix} (c_1, \lambda_1), \ldots, (c_p, \lambda_p) \\ (1 - c, \gamma), (d_1, \mu_1), \ldots, (d_q, \mu_q) \end{matrix} \right] \hat{f}_2(y)\, dy \qquad \ldots(19)$$

where

$$\mathcal{M}\{\hat{f}_2(y); -s\} = \mathcal{M}\{\hat{f}(y); -s\}/\Gamma(1 - c + \gamma s). \qquad \ldots(20)$$

On the other hand, if

$$Q_2(s) = Q(s)\,\{\Gamma(1 - c + s)\}^N \qquad \ldots(21)$$

then

$$f(x) = \int_0^\infty y^{-1} H_{p,q+N}^{q-m+N,p-n}\left[xy \,\middle|\, \begin{matrix} (c_1, \lambda_1), \ldots, (c_p, \lambda_p) \\ (1 - c, 1), \ldots, (1 - c, 1), (d_1, \mu_1), \ldots, (d_q, \mu_q) \end{matrix} \right] \times \hat{f}_3(y)\, dy \qquad \ldots(22)$$

where

$$\mathcal{M}\{\hat{f}_3(y); -s\} = \mathcal{M}\{\hat{f}(y); -s\} \,.\, \{\Gamma(1 - c + s)\}^{-N}. \qquad \ldots(23)$$

Following a similar line of thought, we may suppose that $0 < K < 2$ and note the known relation (Erdélyi *et al.* 1954, p. 349, (15))

$$\mathcal{M}\{x^b(1 + x)^{-a-b}; s\} = \Gamma(b + s)\,\Gamma(a - s)/\Gamma(a + b) \qquad \ldots(24)$$

valid for $\mathrm{Re}\,(a + b) > 0$ and $-\mathrm{Re}\,(b) < \mathrm{Re}\,(s) < \mathrm{Re}\,(a)$. Thus we can multiply each side of equation (11) by $\Gamma(b + s)\,\Gamma(a - s)$ to get

$$\left.\begin{aligned} &\Gamma(a + b)\,\mathcal{M}\{x^b(1 + x)^{-a-b}; s\}\,\mathcal{M}\{f(x); s\} \\ &\quad = Q_3(s)\,\mathcal{M}\{\hat{f}(y); -s\} \end{aligned}\right\} \qquad \ldots(25)$$

where the new quotient $Q_3(s)$, for which $K_3 = 2 - K > 0$, is given by

$$Q_3(s) = Q(s)\,\Gamma(b + s)\,\Gamma(a - s). \qquad \ldots(26)$$

Now the two sides of equation (25) may be inverted by using, respectively, equations (14) and (13) of Erdélyi *et al.* (1954, p. 308) with $\alpha = \beta + 1 = 0$. By an elementary change of variable in the resulting integral on the left-hand side, this will yield

$$x^b\,\Gamma(a+b)\int_x^\infty y^{-a-b}\,(y-x)^{a-1}\,f(y-x)\,dy$$

$$=\int_0^\infty y^{-1}H^{q-m+1,p-n+1}_{p+1,q+1}\left[xy\,\middle|\,\begin{matrix}(1-a,1),(c_1,\lambda_1),\ldots,(c_p,\lambda_p)\\(b,1),(d_1,\mu_1),\ldots,(d_q,\mu_q)\end{matrix}\right]\hat{f}(y)\,dy. \qquad \ldots(27)$$

Analogous, but more complicated, expressions can be obtained for $2N < K < 2N+2$ merely by iteration.

§4. Finally, the use of fractional integrals can be pursued, but only in special cases; it serves more for a reduction technique in these cases than for an inversion formula (cf. also Saxena 1966). The Erdélyi-Kober operators of fractional integration are defined for $\alpha > 0$, $\beta > -\frac{1}{2}$ by

$$\mathcal{K}(\alpha,\beta:r)\,h(x)=[r/\Gamma(\alpha)]x^\beta\int_x^\infty u^{-\beta-r\alpha+r-1}\,(u^r-x^r)^{\alpha-1}\,h(u)\,du \qquad \ldots(28)$$

and

$$\mathcal{J}(\alpha,\beta:r)\,h(x)=[r/\Gamma(\alpha)]\,x^{-\beta-r\alpha+r-1}\int_0^x u^\beta(x^r-u^r)^{\alpha-1}\,h(u)\,du. \qquad \ldots(29)$$

First suppose that $\alpha_1=\beta_q$, $0\leqq m\leqq q-1$, $1\leqq n\leqq p$; recalling that we can rearrange within the products of Γ-functions, this gives us a standard form to use. If we now write

$$Q(s)=Q_4(s)\,\Gamma(1-b_q+\beta_q s)/\Gamma(1-a_1+\alpha_1 s) \qquad \ldots(30)$$

it should be noted that, with $\alpha_1=\beta_q$, the K's associated with $Q(s)$ and $Q_4(s)$ are equal so there is nothing gained from that standpoint. However, this displayed ratio of Γ-functions corresponds to the fractional integral operator

$$\mathcal{J}(a_1-b_q,(1-a_1-\beta_q)/\beta_q:1/\beta_q) \qquad \ldots(31)$$

it being assumed that $\alpha_1=\beta_q$.

The simplification perhaps is better illustrated by returning to the original integral transformation (3) which in this case can be rewritten

$$\hat{f}(y)=\mathcal{J}(a_1-b_q,(1-a_1-\beta_q)/\beta_q:1/\beta_q)\int_0^\infty x^{-1}$$

$$\times H^{m,n-1}_{p-1,q-1}\left[xy\,\middle|\,\begin{matrix}(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_{q-1},\beta_{q-1})\end{matrix}\right]f(x)\,dx. \qquad \ldots(32)$$

The second case arises from $\beta_1 = \alpha_p$, $1 \leqq m \leqq q$, $0 \leqq n \leqq p-1$ in an analogous manner and leads to

$$\hat{f}(y) = \mathcal{K}(a_p - b_1, b_1/\alpha_p : 1/\alpha_p) \times \int_0^\infty x^{-1} H^{m-1,n}_{p-1,q-1}\left[xy \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_{p-1}, \alpha_{p-1}) \\ (b_2, \beta_2), \ldots, (b_q, \beta_q) \end{matrix} \right] f(x)\, dx. \quad \ldots(33)$$

In the special case when all of the α's and β's are equal to 1, the H-function reduces to the G-function of Meijer and the reduction can always be carried out, and repeatedly so, until two of the products of Γ-functions in the Mellin transform of G are empty. Since $p + q < 2(m + n)$, we have three cases. If $q \leqq m + n \leqq p$, then two sets of factors of $1/Q(-s)$ can be separated to introduce the appropriate fractional integrals leaving the kernel of the integral transformation in the form

$$G^{0,n-q}_{p-q,0}\,.$$

Analogously, for $p \leqq m + n \leqq q$, we can obtain the reduced kernel

$$G^{m-p,0}_{0,q-p}.$$

In the third case, when $p \leqq m + n$ and $q \leqq m + n$, we can pair off each of the denominator factors with one from the numerator thereby leaving a kernel of the form

$$G^{m,n}_{n,m}\,.$$

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