ANALYSIS OF STRESS-WAVE PENETRATION IN PLATES—II

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Expressions for mass, momentum and velocity of the scab produced in a plate due to explosive loading are obtained analytically using the solution developed in a previous paper. A numerical example is given.

INTRODUCTION

In an earlier paper (Viswanathan and Biswas 1970, henceforth referred to as part I) the problem of scabbing in an elastic plate under explosive loading was considered. The loading was assumed over a circular area on one face of the plate and equations were obtained giving the contour of the scab. In the present paper we use the approximate solutions obtained in Part I to deduce the expressions for the mass, momentum and velocity of the scab.

AXIAL DISPLACEMENT AND VELOCITY DISTRIBUTION IN THE MEDIUM

The Laplace transform of ω, the z-component of the displacement normal to the plate (cf: Viswanathan and Biswas 1970), is given by

$$p^2\bar{\omega} = -\int_0^\infty C_1^2\, \alpha A(\xi)\, J_0(r\xi)\, e^{-\alpha z}\, d\xi$$

$$+\int_0^\infty \frac{C_1^2\, \alpha A(\xi)\, E_1(\xi)\, J_0(r\xi)\, e^{-\alpha(2H-z)}}{M(\xi)}\, d\xi$$

$$-\int_0^\infty \frac{C_2^2\, \xi A(\xi)\, E_3(\xi)\, J_0(r\xi)\, e^{-\beta(H-z)-\alpha H}}{M(\xi)}\, d\xi \quad \text{.. (1)}$$

where

$$\left.\begin{aligned} \alpha &= \sqrt{\xi^2 + p^2/C_1^2}, \quad \beta = \sqrt{\xi^2 + p^2/C_2^2} \\ A(\xi) &= -\frac{ap^2\bar{f}(p)\, J_1(a\xi)\, (2\xi^2 + p^2/C_2^2)}{\mu C_1^2\, M(\xi)} \end{aligned}\right\} \quad \text{...(2a)}$$

$$\left.\begin{aligned} E_1(\xi) &= -\ \{(2\xi^2 + p^2/C_2^2)^2 + 4\xi^2\alpha\beta\} \\ E_3(\xi) &= -\ 4\frac{C_1^2}{C_2^2}\,\alpha\xi(2\xi^2 + p^2/C_2^2) \\ M(\xi) &= (2\xi^2 + p^2/C_2^2)^2 - 4\,\xi^2\alpha\beta \end{aligned}\right\} \quad \ldots(2b)$$

p is the Laplace transform parameter with respect to time, H the thickness of the plate, a the radius of the loaded region, C_1, C_2 the velocities of compressional and shear waves and $f(t)$ the load. In this paper notations of Part I have been retained for quantities common with Part I.

In (1) the first term is due to the direct compressional (dilatation) function $\triangle_0$, while the second and third terms arise from the reflected compressional and rotational functions $\triangle_1$ and Ω_1 corresponding to $\triangle_0$. For our final results we have retained only the first set of arrivals neglecting others because of their insignificant contributions.

The various wave fronts associated with these functions have been described and discussed in Part I. Following a procedure similar to that for the calculation of the stress components $\bar{\sigma}_z$ and $\bar{\sigma}_r$ of Part I, we obtain the saddle point approximations for $\bar{\omega}$ from (1) to be

$$\bar{\omega} = \bar{\omega}_{S_1+S_1'+S_2+S_2'+S_2''} \quad \ldots(3)$$

where

$$\bar{\omega}_{S_1} = \bar{f}(p)\,\frac{C_1}{p(\lambda + 2\mu)}\,.\,e^{-pz/c_1}\,U(a-r) \quad \ldots(3a)$$

$$\bar{\omega}_{S_1'} = \bar{\omega}_{S_1} \text{ with } z \text{ replaced by } 2H - z \quad \ldots(3b)$$

($U(x) = 1$ for $x > 0$, $= 0$ for $x < 0$) are the exact contributions from the direct compressional plane front S_1 and its reflected tensile plane front S_1', while the direct compressional edge wave S_2 and its reflected compressional and shear components S_2' and S_2'' respectively from $z = H$ lead to (assuming $r < a$ but not too close to a)

$$\bar{\omega}_{S_2} \approx -\frac{\bar{f}(p)}{p^{3/2}}\{\chi_{11}\,e^{-pR_1/C_1} + \chi_{12}\,e^{-pR_2/C_1 + i\pi/2}\},\ \text{for } r \text{ not small}$$

$$\approx -\frac{\bar{f}(p)}{p}\,\chi_{13}\,I_0\left[\frac{p}{C_1}\cdot\frac{ar}{(a^2+z^2)^{1/2}}\right]e^{-(p/C_1)\sqrt{a^2+z^2}},\ \text{for } r \text{ small} \quad \ldots(3c)$$

$$\bar{\omega}_{S_2'} \approx \bar{\omega}_{S_2} \qquad \text{...(3d)}$$

with z replaced by $2H - z$ and χ_{1i}'s are replaced by χ_{2i}'s $(i = 1, 2, 3)$

$$\bar{\omega}_{S_2''} \approx -\frac{\bar{f}(p)}{p^{3/2}}\Big[\chi_{31}\exp\Big\{-p\Big[\frac{\sin\Theta_{12}(H-z)}{C_2}+\frac{\sin\Theta_{11}.\,H}{C_1}+\frac{(a-r)\cos\Theta_{11}}{C_1}\Big]\Big\}+\chi_{32}\exp\Big\{-p\Big[\frac{\sin\Theta_{22}(H-z)}{C_2}+\frac{\sin\Theta_{21}.\,H}{C_1}+\frac{(a+r)\cos\Theta_{21}}{C_1}+\frac{i\pi}{2}\Big]\Big\}\Big], \text{ for } r \text{ not small}$$

$$\approx -\frac{\bar{f}(p)}{p}\chi_{33} I_0\Big(\frac{pr\cos\Theta_{31}}{C_1}\Big)\exp\Big\{-p\Big[\frac{\sin\Theta_{32}(H-z)}{C_2}+\frac{\sin\Theta_{31} H}{C_1}+\frac{a\cos\Theta_{31}}{C_1}\Big]\Big\}, \text{ for } r \text{ small} \qquad \text{...(3e)}$$

Here R's, χ's and Θ's are functions of r and z only, independent of p and have been described later on. The z-component of the velocity $\dot{\omega}$ at each point is given by the Laplace inversion of $p\bar{\omega}$ by the formula

$$\dot{\omega} = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} (p\bar{\omega})\, e^{pt}\, dp. \qquad \text{...(4)}$$

As an example, if we assume

$$f(t) = P\, e^{-\sigma t}\, U(a-r),\ \sigma > 0 \qquad \text{...(5a)}$$

so that

$$\bar{f}(p) = \frac{P}{p+\sigma}\, U(a-r) \qquad \text{...(5b)}$$

then the z-component of the velocity corresponding to the different wave fronts are given by

$$\dot{\omega}_{S_1} = \frac{P}{\lambda+2\mu}\, e^{-\sigma(t-z/C_1)}\, U(t-z/C_1) \qquad \text{...(6a)}$$

$$\dot{\omega}_{S_1'} = \dot{\omega}_{S_1} \text{ with } z \text{ replaced by } 2H-z \qquad \text{...(6b)}$$

$$\dot{\omega}_{S_2} = -\frac{2P}{(\pi\sigma)^{1/2}}\chi_{11} W\{\sigma^{1/2}(t-T_1)^{1/2}\}\, U(t-T_1) - \frac{2P}{(\pi\sigma)^{1/2}}\chi_{12} W\{\sigma^{1/2}(t-T_2)^{1/2}\}\, U(t-T_2) * \Big(-\frac{1}{\pi t}\Big) \quad \text{for } r \text{ not small}$$

$$= -\frac{P}{2\pi}\chi_{13}\int_0^{2\pi} e^{-\sigma(t-T)}\, U(t-T)\, d\theta \text{ for } r \text{ small} \qquad \text{...(6c)}$$

Here $T = (z^2 + r^2 + a^2 - 2ar\cos\theta)^{1/2}/C_1$ and T_1 and T_2 correspond to T for $\theta = 0$ and π respectively.

$$\dot{\omega}_{S_2'} = \dot{\omega}_{S_2} \quad \text{with } z \text{ replaced by } 2H - z$$

and χ_{1i}'s are replaced by χ_{2i}'s $(i = 1, 2, 3)$ ...(6*d*)

$$\dot{\omega}_{S_2'} = -\frac{2P}{(\pi\sigma)^{1/2}}\,\chi_{31}\,W\{\sigma^{1/2}(t - T_1')^{1/2}\}\,U(t - T_1')$$

$$-\frac{2P}{(\pi\sigma)^{1/2}}\,\chi_{32}\,W\{\sigma^{1/2}(t - T_2')^{1/2}\}\,U(t - T_2') * \left(-\frac{1}{\pi t}\right) \quad \text{for } r \text{ not small}$$

$$= -\frac{P}{2\pi}\,\chi_{33}\int_0^{2\pi} e^{-\sigma(t-T')}\,U(t - T')\,d\theta, \qquad \text{for } r \text{ small} \qquad ...(6e)$$

T' is given by

$$T' = \frac{H\,\operatorname{cosec}\Theta_1}{C_1} + \frac{(H - z)\operatorname{cosec}\Theta_2}{C_2}$$

Θ_1, Θ_2 being the solutions of

$$\left.\begin{aligned} \frac{\cos\Theta_1}{C_1} &= \frac{\cos\Theta_2}{C_2} \\ H\cot\Theta_1 + (H - z)\cot\Theta_2 &= (a^2 + r^2 - 2ar\cos\theta)^{1/2} \end{aligned}\right\} \qquad ...(7)$$

and T_1' and T_2' correspond to T' for $\theta = 0$ and π respectively. In the above expressions (*) has been used to denote convolution defined as

$$f(t) * g(t) = \int_{-\infty}^{\infty} g(t - \tau)\,f(\tau)\,d\tau$$

and $W(x)$ is the function $e^{-x^2}\int_0^\infty e^{y^2}\,dy$. For defining χ's we reproduce χ_1 and χ_3 from Part I. These are

$$\chi_1 = \frac{\left(\dfrac{aC}{2\pi R_1\,|\,r\,|}\right)^{1/2}\cdot\left(\dfrac{z}{a-r}\right)\left\{1 - \dfrac{2C_2^2}{C_1^2}\left(\dfrac{a-r}{R_1}\right)^2\right\}^2}{\left\{1 - \dfrac{2C_2^2}{C_1^2}\left(\dfrac{a-r}{R_1}\right)^2\right\}^2 + \dfrac{4C_2^2\,z(a-r)^2}{C_1^3\,R_1^3}\left\{1 - \dfrac{C_2^2}{C_1^2}\left(\dfrac{a-r}{R_1}\right)^2\right\}^{1/2}} \qquad ...(8)$$

$$\chi_3 = \frac{\dfrac{z}{(a^2+z^2)^{1/2}}\cdot\left(1-\dfrac{2C_2^2}{C_1^2}\cdot\dfrac{a^2}{a^2+z^2}\right)^2}{\left(1-\dfrac{2C_2^2}{C_1^2}\;\dfrac{a^2}{a^2+z^2}\right)^2+\dfrac{4C_2^2\,a^2\,z}{C_1^3(a^2+z^2)^{3/2}}\cdot\left(1-\dfrac{C_2^2}{C_1^2}\cdot\dfrac{a^2}{a^2+z^2}\right)^{1/2}} \qquad ...(9)$$

Then

$$\chi_{11} = \frac{C_2^2}{\mu C_1}\cdot\frac{z}{R_1}\cdot\chi_1 \qquad ...(10)$$

$\chi_{12} = \chi_{11}$ with r replaced by $-r$ ...(11)

$$\chi_{13} = \frac{C_2^2}{\mu c_1}\cdot\frac{z}{(a^2+z^2)^{1/2}}\cdot\chi_3 \qquad ...(12)$$

$$\chi_{21} = \left[\left\{1-\frac{2C_2^2}{C_1^2}\left(\frac{a-r}{R_3}\right)^2\right\}^2 - \frac{4C_2^3\,(a-r)^2\,(2H-z)}{C_1^3R_3^3}\times\left\{1-\frac{C_2^2}{C_1^2}\left(\frac{a-r}{R_3}\right)\right\}^{1/2}\right]\chi_{11} \text{ with } z \text{ replaced by } 2H-z \qquad ...(13)$$

$\chi_{22} = \chi_{21}$ with r replaced by $-r$ ...(14)

$$\chi_{23} = \left[\left\{1-\frac{2C_2^2}{C_1^2}\;\frac{a^2}{a^2+(2H-z)^2}\right\}^2 - \frac{4C_2^3\,a^2(2H-z)}{C_1^3\{a^2+(2H-z)^2\}^{3/2}}\times\left\{1-\frac{C_2^2}{C_1^2}\;\frac{a^2}{a^2+(2H-z)^2}\right\}^{1/2}\right]\chi_{13} \qquad ...(15)$$

with z replaced by $2H-z$

$$\chi_{31} = \frac{2}{\mu}\cdot\frac{C_2^4}{C_1^2}\left(\frac{2a}{\pi C_1\,|\,r\,|}\right)^{1/2}\cdot\frac{\cos^2 2\Theta_2\,\sin\Theta_1\,\cos\Theta_1}{\cos^2 2\Theta_2+\dfrac{C_2^2}{C_1^2}\sin 2\Theta_1\,\sin 2\Theta_2}\times\frac{1}{\left[(H-z)\,\mathrm{cosec}^3\,\Theta_2\times\dfrac{C_2}{C_1}+H\,\mathrm{cosec}^3\,\Theta_1\right]^{1/2}} \qquad ...(16)$$

with $\Theta_1 = \Theta_{11}$ and $\Theta_2 = \Theta_{12}$, the solutions of eqns. (7) when $\theta = 0$

$$\chi_{32} = \chi_{31} \qquad ...(17)$$

with $\Theta_1 = \Theta_{21}$ and $\Theta_2 = \Theta_{22}$, the solutions of equations (7) when $\theta = \pi$

$$\chi_{33} = \left(\frac{2\pi\,|\,r\,|}{C_1}\right)^{1/2}\chi_{31} \qquad ...(18)$$

with $\Theta_1 = \Theta_{31}$ and $\Theta_2 = \Theta_{32}$, the solutions of eqns. (7) when $r = 0$ and

$$R = (z^2+a^2+r^2-2ar\cos\theta)^{1/2} \qquad ...(19a)$$

$$R_1 = R \text{ when } \theta = 0 \qquad ...(19b)$$

$$R_2 = R \text{ when } \theta = \pi \qquad \text{...(19c)}$$

$$R_3 = R \text{ when } \theta = 0 \text{ and } z \text{ is replaced by } 2H - z \qquad \text{...(19d)}$$

$$R_4 = R \text{ when } \theta = \pi \text{ and } z \text{ is replaced by } 2H - z. \qquad \text{...(19e)}$$

Mass, Momentum and Velocity of the Scab

We may recall that the equations of the scab surface are [cf: Viswanathan and Biswas 1970, equation (27)]

$$\frac{\Sigma}{P} = 1 - e^{-(2\sigma/C_1)(H-z)} + \frac{2\chi_1}{(\pi\sigma)^{1/2}} \cdot W\left\{\frac{\sigma^{1/2}}{(C_1)^{1/2}}\left[2H - Z - R_1\right]^{1/2}\right\},$$

for r not small

$$\frac{\Sigma}{P} = 1 - e^{-(2\sigma/C_1)(H-z)} + \frac{\chi_3}{2\pi}\int_0^{2\pi} e^{-\sigma T}\, U(T)\, d\theta, \text{ for } r \text{ small} \qquad \text{...(20)}$$

where $T = (2H - z - R)/C_1$

while the scab thickness d is given by

$$d = -\frac{C_1}{2\sigma} \log\left(1 - \frac{\Sigma}{\rho}\right) \qquad \text{...(21)}$$

for the condition $a > H + d$ and $2(Hd)^{1/2} \leqslant a \leqslant H + d$ or by

$$\frac{\Sigma}{\rho} = 1 - e^{-2\sigma d/C_1} + \Big[\chi_3\Big]_{z=H-d} \exp\left\{-\frac{\sigma}{C_1}\left[H + d - \left\{a^2 + (H-d)^2\right\}^{1/2}\right]\right\}$$

...(22)

for the condition $a \leqslant 2\,(Hd)^{1/2}$.

Therefore, the mass of the scab is given by

$$\text{Mass} = \pi\rho \int_{H-d}^{H} \{r(z)\}^2\, dz \qquad \text{...(23)}$$

and momentum of the scab is given by

$$\text{Momentum} = 2\pi\rho \int_{H-d}^{H}\int_0^{r(z)} \dot{\omega}\, r'\, dr'\, dz \qquad \text{...(24)}$$

where $r(z)$ is the solution of equation (20). Finally the velocity of flight of the scab is given by

$$\text{Velocity} = \text{Momentum/Mass}. \qquad \text{...(25)}$$

The integrals (23) and (24) will be functions of t and we will be interested in their values at $t = (H + d)/C_1$, the moment of occurrence of the innermost fracture.

Numerical Results

Given the explosion parameters P, σ and a and the plate parameters H, Σ, λ, μ, ρ the scab thickness d is calculated from (21) and (22) and the lateral surface of the scab is calculated from (20). Once these quantities are known, the mass, momentum and velocity of the scab are easily determined from eqns. (23), (24) and (25) respectively.

A programme in FORTRAN IV language was written for the job outlined above and run on a IBM 360/44 Computer. The numerical results are given below :

Model assumed (*Steel*) (cf: Rinehart and Pearson 1963)

$$\rho = 0.283 \text{ lb/in}^3$$

$$C_1 = 19500 \text{ ft/sec.}$$

$$C_2 = 10200 \text{ ft/sec.}$$

$$\Sigma = 56800 \text{ psi}$$

and

$$P = 3\Sigma$$

$$H = 3 \text{ in}$$

$$a = 4 \text{ in.}$$

Results obtained

$$d = 0.75 \text{ in}$$

Mass of the scab = 3.5 lb

Momentum of the scab = 920.5 lb ft/sec.

Velocity of flight = 263 ft/sec.

The velocity obtained is remarkably close to the observed values in several experiments.

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