ON FOUR-PART MIXED BOUNDARY VALUE PROBLEM OF ELASTICITY

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(Communicated by B. R. Seth, F.N.A.)

(*Received* 28 *May* 1973)

In this paper the author considers the plane strain problem of determining the stress distribution in the vicinity of a Griffith crack in an infinite elastic solid. The crack is opened by two symmetrical rigid inclusions. The geometrical shapes of the inclusion are such as to permit the frictionless contact with the surrounding medium. Depending upon the shapes of the inclusions he obtains quadruple integral equations. A closed form solution of the quadruple integral equations is obtained. Finally, various quantities of physical interest are determined.

1. INTRODUCTION

Recently, three-part mixed boundary value problem is discussed by Lowengrub and Srivastav (1970). In that problem they considered the stress distribution in the neighbourhood of a Griffith crack which is opened by the insertion of one rigid and two symmetrical rigid inclusions. In the case of one rigid inclusion Lowengrub and Srivastav (1970) reduce the problem to the solution of triple trigonometric integral equations whose closed form solution is obtained. In the case of two symmetric rigid inclusions they reduce the solution of the problem to dual integral equations.

In this paper we shall discuss a more general problem than that treated by Lowengrub and Srivastav (1970). We consider the plane strain problem of determining the stress distribution in the vicinity of a Griffith crack in an infinite solid, when two rigid inclusions are inserted in a crack. The displacement is now prescribed over parts of the crack and its remaining boundary being stress-free. As a result we obtain a four-part mixed boundary value problem, which is reduced to the solution of quadruple integral equations. Closed form solution of the quadruple integral equations is obtained. We believe this exact solution is the first of its kind. In the following analysis no attempt is made to justify the change of order of integrations.

2. THE BOUNDARY VALUE PROBLEM AND RELEVANT QUADRUPLE INTEGRAL EQUATIONS

A crack is opened by insertion of two symmetric rigid inclusions which induce normal displacement give $u_y = f(\mid x \mid)$ for $a \leqslant \mid x \mid \leqslant b, y = 0$. The crack lies on the $\mid x \mid < 1, y = 0$. Here it is assumed that the parts of the crack $b < \mid x \mid < 1$, $0 \leqslant \mid x \mid < a, y = 0$ are stress-free. It is also assumed that the geometrical shapes of the inclusions and its material properties are such as to permit a frictionless contact with the surrounding medium. The boundary conditions of the problem are that

$$\sigma_{yy}(x, 0) = 0, \quad 0 \leqslant |x| < a, \quad \sigma_{yy}(x, 0) = 0, \quad b < |x| < 1,$$
$$u_y(x, 0) = f(|x|), \quad a \leqslant |x| \leqslant b, \quad u_y(x, 0) = 0$$

for $|x| > 1$ and $\sigma_{xy}(x, 0) = 0$ for all x. We obtain with the help of Sneddon and Lowengrub (1969 pp. 25-26) that the equation of elastic equilibrium and the condition of vanishing shearing stress for $y = 0$ is satisfied by the well-known solution

$$u_x(x, y) = -\frac{1}{2(1-n)} \int_0^\infty (1 - 2n - \xi y)\, \psi(\xi)\, e^{-\xi y} \sin \xi x \, d\xi \qquad \ldots(1)$$

$$u_y(x, y) = \frac{1}{2(1-n)} \int_0^\infty (2 - 2n - \xi y)\, \psi(\xi)\, e^{-\xi y} \cos \xi x \, d\xi. \qquad \ldots(2)$$

For $y = 0$, the normal components of stress and displacement are given by

$$\sigma_{yy}(x, 0) = -\frac{1}{(1-n)} \int_0^\infty \xi \psi(\xi) \cos \xi x \, d\xi \qquad \ldots(3)$$

$$u_y(x, 0) = \int_0^\infty \psi(\xi) \cos \xi x \, d\xi \qquad \ldots(4)$$

where n is the poisson's ratio of the material. The remaining boundary conditions are satisfied provided that $\psi(x)$ is determined by the quadruple integral equations

$$\int_0^\infty \xi \psi(\xi) \cos \xi x \, d\xi = 0, \qquad 0 < x < a \qquad \ldots(5)$$

$$\int_0^\infty \psi(\xi) \cos \xi x \, d\xi = f(x), \qquad a < x < b \qquad \ldots(6)$$

$$\int_0^\infty \xi \psi(\xi) \cos \xi x \, d\xi = 0, \qquad b < x < 1 \qquad \ldots(7)$$

$$\int_0^\infty \psi(\xi) \cos \xi x \, d\xi = 0, \qquad 1 < x. \qquad \ldots(8)$$

3. The Solution of the Quadruple Integral Equations and Important Physical Quantities

Equation (8) is identically satisfied if we take from Sneddon (1966, pp. 103-4)

$$\psi(u) = \int_0^1 g(t)\, J_0(tu)\, dt \qquad \text{...(9)}$$

where $g(t)$ is to be determined. If we substitute the value of $\psi(u)$ from (9) in (5), (6) and (7) and interchange the order of integrations and make use of the relations from Sneddon [1966, pp. 27-28, (2.1.13), (2.1.14)] we find the triple integral equations

$$\frac{d}{dx} \int_0^x \frac{g(t)\, dt}{\sqrt{x^2 - t^2}} = 0, \qquad 0 < x < a \qquad \text{...(10)}$$

$$\int_x^1 \frac{g(t)\, dt}{\sqrt{t^2 - x^2}} = f(x), \qquad a < x < b \qquad \text{...(11)}$$

$$\frac{d}{dx} \int_0^x \frac{g(t)\, dt}{\sqrt{x^2 - t^2}} = 0, \qquad b < x < 1. \qquad \text{...(12)}$$

We now assume that

$$\frac{d}{dx} \cdot \int_0^x \frac{g(t)\, dt}{\sqrt{x^2 - t^2}} = R(x^2), \qquad a < x < 1 \qquad \text{...(13)}$$

where $R(x^2)$ is to be determined. Solving the equations (10) and (13) with the help of the solution of Abel's type integral equation we get

$$g(t) = \frac{2t}{\pi} \int_a^t \frac{R(x^2)\, dx}{\sqrt{t^2 - x^2}}, \qquad a < t < 1. \qquad \text{...(14)}$$

Substituting the value of $g(t)$ from (14) in (11) and interchanging the order of integrations we get

$$\frac{1}{\pi} \int_a^1 R(y^2) \log \left| \frac{\sqrt{1 - y^2} + \sqrt{1 - x^2}}{\sqrt{1 - y^2} - \sqrt{1 - x^2}} \right| dy = f(x), \quad a < x < b. \qquad \text{...(15)}$$

We have with the help of the relations (12) and (13)

$$R(y^2) = \begin{cases} h(y^2), & a < y < b \\ 0, & b < y < 1 \end{cases} \qquad \text{...(16)}$$

where $h(y^2)$ is to be determined. Now (15) can be written in the form

$$\frac{1}{\pi}\int_a^b h(y^2)\log\left|\frac{\dfrac{1}{\sqrt{1-y^2}}+\dfrac{1}{\sqrt{1-x^2}}}{\dfrac{1}{\sqrt{1-y^2}}-\dfrac{1}{\sqrt{1-x^2}}}\right| dy = f(x),\quad a < x < b. \quad \ldots(17)$$

Using the result of Parihar (1971, Lemma 2, p. 257) we get the solution of the integral equation (17) in the following form :

$$h(y^2) = \frac{2y\sqrt{1-y^2}}{\pi\sqrt{(y^2-a^2)(b^2-y^2)}}\int_a^b \frac{\sqrt{(x^2-a^2)(b^2-x^2)}\,f'(x)\,dx}{(y^2-x^2)\sqrt{1-x^2}}$$

$$+\frac{B\sqrt{1-a^2}}{2F\left[\dfrac{\pi}{2},\sqrt{\dfrac{b^2-a^2}{1-a^2}}\right]}\frac{y}{\sqrt{(1-y^2)(y^2-a^2)(b^2-y^2)}},\quad a < y < b \quad \ldots(18)$$

where

$$B = \frac{2\sqrt{1-a^2}}{F\left[\dfrac{\pi}{2},\sqrt{\dfrac{1-b^2}{1-a^2}}\right]}\int_a^b \frac{xf(x)\,dx}{\sqrt{(1-x^2)(x^2-a^2)(b^2-x^2)}}$$

$$-\frac{4}{\pi}\int_a^b \frac{y\sqrt{1-y^2}\,dy}{\sqrt{(y^2-a^2)(b^2-y^2)}}\int_a^b \frac{\sqrt{(x^2-a^2)(b^2-x^2)}\,f'(x)\,dx}{(y^2-x^2)\sqrt{1-x^2}}. \quad \ldots(19)$$

Here $f'(x)$ denotes differentiation with respect to x and F is the elliptic integral of first kind. Knowing $h(y^2)$ with the help of (18) and (19) we can get $R(y^2)$ from (16) and $g(t)$ from (14) and hence $\psi(u)$ from (9).

The normal stress $\sigma_{yy}(x, 0)$ is given by the relation, for $a < x < b$

$$\sigma_{yy}(x, 0) = -\frac{h(x^2)}{(1-n)} \quad \ldots(20)$$

and that, for $x > 1$

$$\sigma_{yy}(x, 0) = -\frac{1}{(1-n)}\frac{d}{dx}\int_0^1 \frac{g(t)\,dt}{\sqrt{x^2-t^2}}. \quad \ldots(21)$$

Making use of (10), (13) and (16) we can write the expression for $g(t)$ in the following form:

$$g(t) = \frac{2t}{\pi}\int_0^t \frac{G(u^2)\,du}{\sqrt{t^2-u^2}},\quad 0 < t < 1 \quad \ldots(22)$$

where

$$G(u^2) = \begin{cases} 0, & 0 < u < a \\ h(u^2), & a < u < b \\ 0, & b < u < 1. \end{cases} \qquad \ldots(23)$$

Substituting the expression for $g(t)$ from (22) in (21) and interchanging the order of integrations we find

$$\sigma_{yy}(x, 0) = \frac{2x}{\pi\,(1-n)\sqrt{x^2-1}} \int_0^x \frac{G(u^2)\sqrt{1-u^2}}{x^2-u^2}\,du, \; x > 1.$$

With the help of (23), the above expression can be written in the form

$$\sigma_{yy}(x, 0) = \frac{2x}{\pi\,(1-n)\sqrt{x^2-1}} \int_a^b \frac{h(u^2)\sqrt{1-u^2}\,du}{x^2-u^2}, \; x > 1. \qquad \ldots(24)$$

The stress intensity factors at $x = a, b, 1$ are given by

$$K_a = \lim_{x \to a^+} -\sqrt{x-a}\left[\frac{h(x^2)}{1-n}\right] \qquad \ldots(25)$$

$$K_b = \lim_{x \to b^-} -\sqrt{b-x}\left[\frac{h(x^2)}{1-n}\right] \qquad \ldots(26)$$

$$K_c = \lim_{x \to 1^+} \frac{2x\sqrt{x-1}}{\pi(1-n)\sqrt{x^2-1}} \int_a^b \frac{h(u^2)\sqrt{1-u^2}}{x^2-u^2}\,du. \qquad \ldots(27)$$

A Particular Case

Let us consider the case $f(x) = \delta\sqrt{1-x^2}$ for $a \leqslant x \leqslant b$, where δ is a constant. Substituting the values of $f(x)$ and $f'(x)$ in (18) and (19), we find that

$$h(x^2) = \frac{\delta x\sqrt{1-x^2}}{\sqrt{(x^2-a^2)(b^2-x^2)}} - \frac{\delta x\sqrt{(1-a^2)(1-b^2)}}{\sqrt{(1-x^2)(x^2-a^2)(b^2-x^2)}}$$
$$+ \frac{Bx\sqrt{1-a^2}}{2F\left[\frac{\pi}{2}, \sqrt{\frac{b^2-a^2}{1-a^2}}\right]\sqrt{(1-x^2)(x^2-a^2)(b^2-x^2)}}, \; a < x < b \qquad \ldots(28)$$

where

$$B = \frac{\pi\delta\sqrt{1-a^2}}{F\left[\frac{\pi}{2}, \sqrt{\frac{1-b^2}{1-a^2}}\right]} + 2\delta\sqrt{1-b^2}\,F\left[\frac{\pi}{2}, \sqrt{\frac{b^2-a^2}{1-a^2}}\right]$$
$$- 2\delta\sqrt{1-a^2}\,E\left[\frac{\pi}{2}, \sqrt{\frac{b^2-a^2}{1-a^2}}\right], \qquad \ldots(29)$$

and E is the elliptic integral of second kind. Making use of (28) we get from (25) and (26)

$$K_a = \frac{\delta}{(1-n)}\sqrt{\frac{a}{2(b^2-a^2)}}\left[\sqrt{1-b^2}-\sqrt{1-a^2}-\frac{B}{2\delta F\left[\frac{\pi}{2},\sqrt{\frac{b^2-a^2}{1-a^2}}\right]}\right] \quad \text{...(30)}$$

$$K_b = \frac{\delta}{(1-n)}\sqrt{\frac{b}{2(b^2-a^2)}}\left[\sqrt{1-a^2}-\sqrt{1-b^2} - \frac{B}{2F\left[\frac{\pi}{2},\sqrt{\frac{b^2-a^2}{1-a^2}}\right]\delta\sqrt{\frac{1-b^2}{1-a^2}}}\right] \quad \text{...(31)}$$

Equation (27) can be written in the form

$$K_c = \frac{\sqrt{2}}{\pi(1-n)}\int_a^b \frac{h(u^2)\,\delta u}{\sqrt{1-u^2}}. \quad \text{...(32)}$$

With the help of (28) we get from (32)

$$K_c = \frac{B}{2\sqrt{2}\,(1-n)\sqrt{1-b^2}\,F\left[\frac{\pi}{2},\sqrt{\frac{b^2-a^2}{1-a^2}}\right]}. \quad \text{...(33)}$$

We might also note that when $b \to 1$, we get the solution of the same problem as given by Lowengrub and Srivastav (1970, Section 3, pp. 891-93). If we take $b = 1$, then we find from (18)

$$h(y^2) = \frac{2y}{\pi\sqrt{y^2-a^2}}\int_a^1 \frac{\sqrt{x^2-a^2}}{y^2-x^2} f'(x)\,dx, \qquad a < y < 1 \quad \text{...(34)}$$

In particular, if $f(x) = \dfrac{\delta(1-x^2)}{2}$, then after some manipulations we find with the help of (20) and (34)

$$\sigma_{yy}(x,0) = -\frac{2\delta}{\pi(1-n)}\left[x\sqrt{\frac{1-a^2}{x^2-a^2}} - x\log\frac{\sqrt{x^2-a^2}+\sqrt{1-a^2}}{\sqrt{1-x^2}}\right],$$
$$a < x < 1. \quad \text{...(35)}$$

The expression (35) is in complete agreement with the expression (3.9) of the paper of Lowengrub and Srivastav (1970, p. 892) except for the constant factor $1/(1-\eta)$. The expressions (3.8), (3.9), (3.10) and (3.11) given by Lowengrub and Srivastav (1970, p. 892) have a misprint which can be corrected by the insertion of the elastic constant. In like manner a minus sign is to be included in their eqns. (2.3) and (3.8).

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