# APPROXIMATE ANALYTIC SOLUTION OF CONVERGING SPHERICAL AND CYLINDRICAL SHOCKS WITH ZERO TEMPERATURE GRADIENT IN THE REAR FLOW FIELD

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The author has employed Chernyii's technique, in which flow variables are expanded in series of powers of β, the density ratio across the strong shock, to study the converging spherical and cylindrical shock when it approaches the focus (axis) and assumes a self-similar character. The flow behind the shock is assumed to be spatially isothermal rather than adiabatic to simulate the conditions of large radiative transfer behind the shock. The solution in closed form up to second order term in β is obtained which is in good agreement with the exact solution. An analytic expression for the similarity exponent has also been obtained which gives values of the similarity exponent very close to axact values.

## 1. INTRODUCTION

In gas dynamics two types of self-similar processes, termed as self-similar motions of the first kind and the second kind (Zel'dovich and Raizer 1967), have been considered. Taylor's explosion problem and one dimensional centred rarefaction waves are examples of the flows of the first kind while emergence of strong shock near the surface of a star, Sakurai (1960), Zel'dovich and Raizer (1967), Sachdev and Ashraf (1971a), and converging cylindrical and spherical shocks (Guderley 1942), are examples of the flows of the second kind. In self-similar solution of the first kind the similarity exponent is determind from the dimensional considerations or the laws of conservation while in the solution of the second kind this exponent cannot be determined from these considerations in advance but is found from solving the differential equations which govern the flow. The solutions of self-similar motion of the second kind are examples of solutions of differential equations which 'partially' forget their 'initial conditions'.

In this paper we consider imploding spherical and cylindrical shocks near the centre (axis) of implosion when the flow assumes a self-similar character. The shock becomes stronger as it converges towards the centre (axis) and there is high temperature behind the shock leading to intense exchange of heat by radiation or conduction.

The flow behind the shock is not adiabatic but is approximately isothermal. The time dependent temperature behind the shock goes on changing as the shock propagates and this temperature is different from that ahead of the shock. The flow behind the shock is likely to have nearly uniform spatial distribution, that is, $\partial T/\partial r$, the temperature gradient, is zero. Such flows, termed as 'homothermal flows' have been dealt with by Korobeinikov (1956) and Korobeinikov and Riazanov (1959). Sachdev and Ashraf (1971) have considered the homothermal flows behind a strong shock near the edge of a star. Zababakhin and Simonenko (1965) have found that the imploding shock becomes nearly isothermal as it approaches its focus. Except for the idealised intense heat exchange behind the shock, the problem is the same as has been discussed by Gurderly. Recently, Sachdev and Ashraf (1971b) have solved the problem numerically.

In this paper we have found an approximate analytic solution of this problem by employing Chernyii's technique, (1957, 1960, 1961) in which the flow variables are expanded in series of powers of $\beta$, the density ratio across the strong shock. We have found the solution in closed form up to second order terms in $\beta$. Usually the similarity exponent, occurring in the law of shock propagation, is obtained by solving an eigen-value problem for a single differential equation to which the similarity equations are reducible. We have found an analytic expression for $\alpha$, the similarity exponent, from the considerations of singular points of the single differential equation and this gives values of $\alpha$ very close to the exact values. While the values of $\alpha$ obtained from Whitham's rule differ considerably from the exact values (Sachdev and Ashraf 1971b) our analytic expression for the similarity exponent gives values which differ from the exact values by less than 4 per cent. Our numerical results are in very close agreement with the exact solution. The error in solution is $O(\beta^3)$ which is very small when $\gamma$ is $O(1)$.

## 2.1. Basic Equations and Boundary Conditions

We take time $t$ to be negative before the shock converges to the centre (axis) of symmetry and $t = 0$ is the instant at which the shock converges to the centre (axis). The shock position is assumed to be given by

$$R = A(-t)^{\alpha} \qquad \ldots(2.1)$$

where $R$ is the radial distance of the shock from the centre (axis) and $A$ and $\alpha$ are positive constants

The basic equations governing the one dimensional flow in terms of Lagrangian coordinate $\eta$ and time $t$, having temperature as function of time only are

$$\frac{\partial r}{\partial \eta} = \frac{1}{\rho r^{j-1}} \qquad \ldots(2.2)$$

$$\frac{\partial u}{\partial t} = - r^{j-1} \frac{\partial p}{\partial \eta} \qquad \ldots(2.3)$$

$$\frac{\partial T}{\partial r} = 0 \qquad \ldots(2.4)$$

where $r$ is the distance of the particle from the centre or axis of symmetry, and $u$, $\rho$, $p$, and $T$ are the particle velocity, density, pressure and temperature behind the shock wave respectively. $\eta$ is the Lagrangian coordinate defined by

$$d\eta = \rho_0 r_0^{j-1} \, dr_0 \qquad \ldots(2.5)$$

where $\rho_0$ is the ambient density, $r_0$ is the value of $r$ at the initial instant of time and $j = 2, 3$ for cylindrical and spherical symmetry respectively.

The equation of continuity (2.2) may be expressed in terms of particle velocity $u$ as

$$\frac{\partial u}{\partial \eta} = - \frac{1}{\rho^2 r^{j-1}} \left[ \frac{\partial \rho}{\partial t} + (j-1) \frac{\rho u}{r} \right]. \qquad \ldots(2.6)$$

We assume that the radiative flux across the optically thin shock layer is continuous so that the classical shock conditions hold. Thus for a strong shock we have the boundary conditions at the shock as

$$\left. \begin{aligned} u_s &= (1 - \beta) \dot{R} \\ \rho_s &= \rho_0 / \beta \\ p_s &= (1 - \beta) \rho_0 \dot{R}^2 \end{aligned} \right\} \qquad \ldots(2.7)$$

where $\rho_0$ is the ambient density, $\dot{R}$ the shock velocity and $\beta$ the density ratio across a strong shock and is equal to $\frac{\gamma - 1}{\gamma + 1}$.

## 2.2. Self-similar Solution

We introduce a similarity variable $\mu = \frac{\eta}{\eta_s}$ and seek a solution in the form

$$r = R \xi(\mu) \qquad \ldots(2.8)$$

$$u = (1 - \beta) \dot{R} v(\mu) \qquad \ldots(2.9)$$

$$\rho = \frac{\rho_0}{\beta} g(\mu) \qquad \ldots(2.10)$$

$$p = (1 - \beta) \rho_0 \dot{R}^2 \pi(\mu) \qquad \ldots(2.11)$$

where $v(\mu)$, $g(\mu)$ and $\pi(\mu)$ are reduced particle velocity, density and pressure respectively and $\eta_s = \frac{\rho_0 R^j}{j}$ is the value of $\eta$ at the shock.

The boundary conditions (2.7) at the shock, where $\mu = 1$, for the reduced functions $v$, $g$ and $\pi$ become

$$v(1) = g(1) = \pi(1) = 1. \qquad ...(2.12)$$

The equation (2.4) reduces to

$$\frac{d}{d\mu}\left(\frac{\pi}{g}\right) = 0. \qquad ...(2.13)$$

Integrating eqn. (2.13) with boundry conditions (2.12) we obtain

$$g(\mu) = \pi(\mu). \qquad ...(2.14)$$

We substitute expressions (2.8) to (2.11) in eqns. (2.2), (2.6) and (2.3) and making use of eqn. (2.14), we get

$$\frac{d\xi(\mu)}{d\mu} = \frac{\beta}{j} \times \frac{1}{g(\mu)\,\xi(\mu)^{j-1}} \qquad ...(2.14a)$$

$$\frac{dv(\mu)}{d\mu} = \frac{\beta}{(1-\beta)\,g^2\xi^{j-1}}\left[\frac{dg}{d\mu} - \frac{(j-1)}{j}(1-\beta)\frac{vg}{\xi}\right] \qquad ...(2.15)$$

$$\frac{dg}{d\mu} = \frac{1}{j\xi^{j-1}}\left[\lambda v + j\mu\frac{dv}{d\mu}\right] \qquad ...(2.16)$$

where

$$\lambda = \alpha^{-1}(1-\alpha).$$

We expand $\xi$, $v$ and $g$ as

$$\xi = \xi^{(0)} + \beta\xi^{(1)} + \beta^2\xi^{(2)} + \ldots \qquad ...(2.17)$$

$$v = v^{(0)} + \beta v^{(1)} + \beta^2 v^{(2)} + \ldots \qquad ...(2.18)$$

$$g = g^{(0)} + \beta g^{(1)} + \beta^2 g^{(2)} + \ldots \qquad ...(2.19)$$

The boundary conditions at the shock are

$$\left.\begin{aligned} \xi^{(0)} &= 1, \ \xi^{(1)} = \xi^{(2)} = \ldots = 0 \\ v^{(0)} &= 1, \ v^{(1)} = v^{(2)} = \ldots = 0 \\ g^{(0)} &= 1, \ g^{(1)} = g^{(2)} = \ldots = 0. \end{aligned}\right\} \qquad ...(2.20)$$

We substitute the expansions (2.17) to (2.19) in eqns. (2.14$a$) to (2.16) and obtain the following equations for the zeroeth, first and second order terms

$$\frac{d\xi^{(0)}}{d\mu} = 0 \qquad ...(2.21)$$

$$\frac{d\nu^{(0)}}{d\mu} = 0 \qquad \ldots(2.22)$$

$$j\,\xi^{(0)j-1}\,\frac{dg^{(0)}}{d\mu} = \nu^{(0)} + j\mu\,\frac{d\nu^{(0)}}{d\mu} \qquad \ldots(2.23)$$

$$\frac{d\xi^{(1)}}{d\mu} = \frac{1}{jg^{(0)}\,\xi^{(0)j-1}} \qquad \ldots(2.24)$$

$$\frac{d\nu^{(1)}}{d\mu} = \frac{d\nu^{(0)}}{d\mu} + \frac{1}{g^{2(0)}\,\xi^{(0)j-1}}\left[\frac{dg^{(0)}}{d\mu} - \frac{(j-1)}{j} \times \frac{\nu^{(0)}\,g^{(0)}}{\xi^{(0)}}\right] \qquad \ldots(2.25)$$

$$j\xi^{(0)j-1}\,\frac{dg^{(1)}}{d\mu} = \lambda\,\nu^{(1)} + j\mu\,\frac{d\nu^{(1)}}{d\mu} - j(j-1)\;\xi^{(0)j-1}\,\xi^{(1)}\,\frac{dg^{(0)}}{d\mu} \qquad \ldots(2.26)$$

$$\frac{d\xi^{(2)}}{d\mu} = \frac{-1}{jg^{(0)}\xi^{(0)j-1}}\left[\frac{g^{(1)}}{g^{(0)}} + (j-1)\,\frac{\xi^{(1)}}{{}^{(0)}}\right] \qquad \ldots(2.27)$$

$$\frac{d\nu^{(2)}}{d\mu} = \frac{d\nu^{(1)}}{d\mu} + \frac{\mu}{g^{2(0)}\xi^{(0)j-1}}\left[\frac{dg^{(1)}}{d\mu} - \frac{2g^{(1)}}{g^{(0)}}\right.$$

$$\left.\times\,\frac{dg^{(0)}}{d\mu} - (j-1)\,\frac{\xi^{(1)}}{\xi^{(0)}}\,\frac{dg^{(0)}}{d\mu}\right]$$

$$-\,\frac{(j-1)}{jg^{(0)}\,\xi^{j(0)}}\left[\nu^{(1)} - \nu^{(0)}\left\{\frac{g^{(1)}}{g^{(0)}} + j\,\frac{\xi^{(1)}}{\xi^{(0)}} + 1\right\}\right] \qquad \ldots(2.28)$$

$$j\xi^{(0)j-1}\,\frac{dg^{(2)}}{d\mu} = \lambda\nu^{(2)} + j\,\frac{d\nu^{(2)}}{d\mu} - j\xi^{(0)j-1}\Big[(j-1)$$

$$\times\,\frac{\xi^{(1)}}{\xi^{(0)}}\,\frac{dg^{(1)}}{d\mu^{(0)}} + (j-1)\,\frac{\xi^{(2)}}{\xi^{(0)}}\,\frac{dg^{(0)}}{d\mu}$$

$$+\,\frac{(j-1)(j-2)}{2}\left(\frac{\xi^{(1)}}{\xi^{(0)}}\right)^2\frac{dg^{(0)}}{d\mu}\Big]. \qquad \ldots(2.29)$$

We integrate eqns. (2.21) to (2.29) with boundary conditions (2.20) and obtain the solutions as

$$\xi^{(0)} = 1 \qquad \ldots(2.30)$$

$$\xi^{(1)} = \frac{1}{\lambda}\ln(\mu + K) - A \qquad \ldots(2.31)$$

$$\xi^{(2)} = \left[A_1 + \frac{B_1}{\mu + K} + C_1\ln(\mu + K)\right]\ln(\mu + K)$$

$$+\,\frac{1}{(\mu + K)^2}\,[E_1 + D_1(\mu + K)] + F_1 \qquad \ldots(2.32)$$

$$\nu^{(0)}(\mu) = 1 \qquad \ldots(2.33)$$

$$v^{(1)}(\mu) = \frac{jk}{\lambda}\left[\frac{1}{\mu + k} - \frac{1}{1 + k}\right] + \ln\left(\frac{\mu + K}{1 + K}\right) \qquad ...(2.34)$$

$$v^{(2)}(\mu) = \frac{1}{(\mu + K)^3}\,[A_2\mu^3 + B_2\mu^2 + C_2\mu + D_2]$$

$$+ (E_2\mu^2 + F_2\mu + G_2)\,\frac{\ln(\mu + K)}{(\mu + K)^2}$$

$$+ H_2\,[\ln(\mu + K)]^2 + L_2 \qquad ...(2.35)$$

$$g^{(0)} = \frac{\lambda}{j}\,(\mu + K) \qquad ...(2.36)$$

$$g^{(1)} = P\mu + \frac{Q}{\mu + K} + (\overline{R} + S\mu)\, ln(\mu + K) + \qquad ...(2.37)$$

$$g^{(2)} = A_3\mu + \frac{1}{(\mu + K)^3}\,[B_3\mu^4 + C_3\mu^3 + D_3\mu^2 + E_3\mu + F_3]$$

$$+ \frac{\ln(\mu + K)}{(\mu + K)^2}\,(G_3\mu^3 + H_3\mu^2 + I_3\mu + J_3) + (K_3 + L_3\mu)$$

$$\times\,[\ln(\mu + K)]^2 + L_3. \qquad ...(2.38)$$

We have omitted explicit expressions for the constants $A_1$, $A_2$ ... for the sake of brevity.

From eqns. (2.15) and (2.16) we obtain

$$\frac{dg}{d\mu} = \frac{(1 - \beta)\, gv}{j\xi}\;\frac{[\lambda g\xi^j - (j - 1)\,\beta]}{(1 - \beta)\, g^2\xi^{2(j-1)} - \mu^2\beta}. \qquad ...(2.39)$$

The differential equation (2.39) possesses several singular points. In order the solution may be physically meaningful, the solution curve, besides satisfying the boundary condition at the shock, passes on through the appropriate singular point. The solution is finite and single valued if both the denominator and the numerator of differential equation (2.39) vanish simultaneously. The denominator and the numerator vanish along the path

$$g\xi^{j-1} = \sqrt{\frac{\beta}{1 - \beta}}\,\mu,\; \xi = 1 \qquad ...(2.40)$$

which gives

$$\alpha = [1 + (j - 1)\sqrt{\beta(1 - \beta)}\,]^{-1}. \qquad ...(2.41)$$

## 3. Results and Discussions

We have obtained approximate analytic solution, given by equation (2.30) to (2.38), in closed form up to second order terms in $\beta$. The error in the solution is of $O(\beta^3)$ which is small when $\gamma$ is $O(1)$.

Equations (2.33) to (2.35) give the reduced velocity distribution. To the zeroeth order approximation the particle velocity is same as the shock velocity while the density and pressure are linear functions of $\mu$. Thus the first and second order terms conribute more significantly to velocity than to the density and pressure. The eqn. (2.36) shows, that the density and pressure increase behind the shock. The first and second order terms in the solution are too complicated to lend themselves to visual estimation. Equations (2.30) to (2.32) give the Eulerian distance $r$ and Eulerian similarity variable $\xi$ as function of Lagrangian similarity variable $\mu$. To the zeroeth approximation the Eulerian distance is same as the shock distance.

Usually the similarity exponent $\alpha$ is obtained by solving the differential equations governing the flow behind the shock. We have obtained an analytic expression for $\alpha$, the equation (2.41), from the considerations of singular points of equation (2.39). This expression for $\alpha$ gives the values of the similarity exponent remarkably close to exact values. While Whitham's rule (1958) gives the values of $\alpha$ which differ from the exact values by more than 10 per cent, the values of $\alpha$ computed from the equation (2.41) differ from the exact values by less than 4 per cent in all cases. Tables I, II and III give values of $\alpha$ obtained from our analytic solution, numerical integration and Whitham's rule respectively. Table IV gives the value of the similarity exponent for the adiabatic flow. We find that $\alpha$ is smaller for the homothermal flows than the adiabatic flows so that the shock velocity $\dot{R} \alpha R^{-(1-\alpha)/\alpha}$ is larger in the former case than in the latter as the shock approaches the centre (or axis). The shock velocity and hence pressure tend to infinity as $t \to 0$, zero being the instant of shock implosion.

TABLE I

*Values of $\alpha$ obtained from the analytic expression*

| | Cylindrical $j = 2$ | Spherical $j = 3$ |
|---|---|---|
| 6/5 | 0.7767 | 0.6349 |
| 7/5 | 0.7285 | 0.5730 |
| 5/3 | 0.6978 | 0.5359 |

TABLE II

*Values of $\alpha$ obtained from numerical integration*

| | Cylindrical | Spherical |
|---|---|---|
| 6/5 | 0.7963 | 0.6553 |
| 7/5 | 0.7530 | 0.5965 |
| 5/3 | 0.7218 | 0.5589 |

Figures 1 and 2 show the distributions of reduced particle velocity, density and pressure behind the shock for cylindrical and spherical shocks respectively. The exact values of $\frac{u}{\dot{R}}$, $\frac{\rho}{\rho_0}$ and $\frac{p}{\rho_0 \dot{R}^2}$ have been shown by dotted curves in the case of

TABLE III

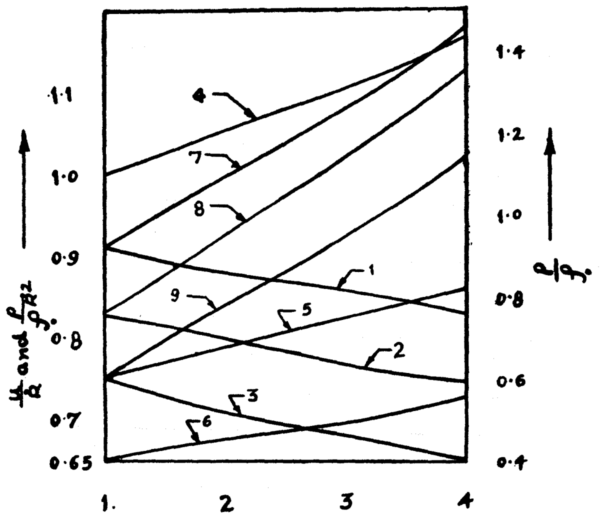
*Values of α obtained from Whitham's rule*

| | Cylindrical | Spherical |
|---|---|---|
| 6/5 | 0.8598 | 0.7540 |
| 7/5 | 0.8354 | 0.7173 |
| 5/3 | 0.8161 | 0.6892 |

TABLE IV

*Values of α for the adiabatic flow*

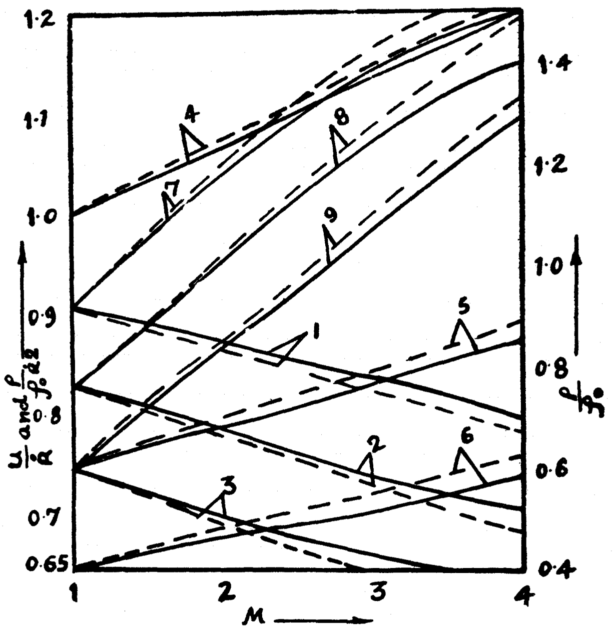
| | Cylindrical | Spherical |
|---|---|---|
| 6/5 | 0.8612 | 0.7572 |
| 7/5 | 0.8352 | 0.7172 |
| 5/3 | 0.8156 | 0.6884 |




$\frac{u}{\dot{R}}$ [1. $\gamma=6/5$, 2. $\gamma=7/5$, 3. $\gamma=5/3$], $\frac{p}{\rho_0}$ [4. $\gamma=6/5$, 5. $\gamma=7/5$, 6. $\gamma=5/3$], $\frac{p}{\rho_0 \dot{R}^2}$ [7. $\gamma=6/5$, 8. $\gamma=7/5$, 9. $\gamma=5/3$]

FIG. 1. Flow variable distributions behind the cylindrical shock.

spherical shock only. Our analytic solution is in excellent agreement with the exact solution. Sachdev and Ashraf (1971a) have compared the homothermal flow with the adiabatic flow and have shown that the total energy $E \sim R^j \dot{R}^2 R^{j+2-(2/\alpha)}$ in the region of self-similar solution decreases as the shock converges to its focus. In the spherical symmetry for $\gamma = 7/5$, the energy $E$ in the adiabatic flow and homothermal flow are $O(R^{2 \cdot 21})$ and $O(R^{3 \cdot 352})$ respectively. Thus, in the homothermal flow, the total energy in the region of self-similar flow decreases more rapidly with time than in the adiabatic flow. Our solution is not valid when $\gamma \to \infty$




$\frac{u}{\dot{R}}$ [1. $\gamma=6/5$, 2. $\gamma=7/5$, 3. $\gamma=5/3$], $\frac{p}{\rho_0}$ [4. $\gamma=6/5$, 5. $\gamma=7/5$, 6. $\gamma=5/3$], $\frac{p}{\rho_0}\dot{R}^2$ [7. $\gamma=6/5$, 8. $\gamma=7/5$, 9. $\gamma=5/3$]

FIG. 2. Flow variable distributions behind the spherical shock.

which corresponds to $t = 0$, $r \neq 0$, the instant of implosion, or $r = \infty$, $t \neq 0$, a point far behind the shock. In fact the self-similar solution, Zel'dovich and Raizer, is valid only in a small region near the centre (axis) of focus, to which all non-self-similar solutions of this problem finally converge and in this region our analytic solution gives very good results.

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