ON THE SUM OF APPELL FUNCTION F_2

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A sum of the terminating F_2 has been obtained and the result has been applied to find two transformations, one of which is known and the other is unknown.

1. INTRODUCTION

The sums of Appell functions at different points have been given from time to time by various mathematicians namely, Pandey (1960), Jain (1966), Carlitz (1967a, b), Lavoie and Trottier (1969) under different conditions of parameters. In the present paper an attempt has been made to obtain the sum of terminating F_2 analogous to F_3 viz.,

$$F_3(c-a, c: -m, -n : c : 1, 1) = \frac{(a)_m (c-a)_n}{(c)_{m+n}}$$

which is obtained from Jain (1966) and Bailey (1964) by making b tend to infinity.

The result has been applied to obtain four transformations of Horn's functions, three of them are believed to be new. It has also been shown that main result is equivalent to a transformation.

2. MAIN RESULT

The result to be proved is

$$F_2(a : -m, -n : 1+a-e, e : 1, 1) = \frac{(e-a)_{n-m}}{(e)_{n-m}} . \qquad \ldots(2.1)$$

PROOF : We have

$$F_2(a : -m, -n : 1+a-e, e : 1, 1) = \sum_{p,q} \frac{(a)_{p+q}(-m)_p(-n)_q}{p!\,q!(1+a-e)_p(e)_q}$$

$$= \sum_{p=0}^{m} \frac{(a)_p(-m)_p}{p!(1+a-e)_p} \cdot {}_2F_1\left(\begin{matrix}-n, a+p\\ e\end{matrix}; 1\right).$$

Now summing ${}_2F_1\left(\begin{matrix}-n, a+p\\ e\end{matrix}; 1\right)$ by Vandermonde's theorem (Bailey 1964, p. 3) we have

$$F_2(a:-m,-n:1+a-e,e:1,1)=\frac{(e-a)_n}{(e)_n}\;{}_2F_1\left(\begin{matrix}-m,a\\1+a-e+n\end{matrix};1\right).$$

Now summing ${}_2F_1\left(\begin{matrix}-m,a\\1+a-e+n\end{matrix};1\right)$ by Vandermonde's theorem once again we get (2.1).

3. Applications

In this section we make use of the sum (2.1) to obtain some transformations of Horn's functions (Erdélyi 1953, p. 224).

The transformations to be proved are

$$G_1(a,c-b,1-c:x,y)=(1+x+y)^{-a}$$
$$\times F_4\left(a,b:1+b-c,c:\frac{x}{1+x+y},\frac{y}{1+x+y}\right) \qquad \text{...(3.1)}$$

$$G_2(a,a',c-b,1-c:x,y)=(1+x)^{-a}(1+y)^{-a'}$$
$$\times F_2\left(b:a,a':1+b-c,c:\frac{x}{1+x},\frac{y}{1+y}\right) \qquad \text{...(3.2)}$$

$$G_3(e-a,1-e:x,y)=(1+x)^{a-1}$$
$$\times G_1\left(1-a,e-a,1-e:\frac{x}{1+x},\frac{x}{1+x}\right) \qquad \text{...(3.3)}$$

and

$$H_2(1-e,e-a,e-a,e,e-a:x,y)$$
$$=(1+y)^{-e}\,G_2\left(1-a,e,a,1-e:-x,\frac{-y}{1+y}\right). \qquad \text{...(3.4)}$$

The transformations (3.1), (3.3) and (3.4) are believed to be new.

To prove (3.1) we proceed as follows:

By definition

$$G_1(a,c-b,1-c:x,y)=\sum_{m,n}\frac{(a)_{m+n}(c-b)_{n-m}(1-c)_{m-n}}{m!\,n!}x^m y^n$$
$$=\sum_{m,n}\frac{(a)_{m+n}}{m!\,n!}(-x)^m(-y)^n\frac{(c-b)_{n-m}}{(c)_{n-m}}.$$

Using (2.1) we have

$$G_1(x,y)=\sum_{m,n}\frac{(a)_{m+n}}{m!\,n!}(-x)^m(-y)^n$$
$$\times F_2(b:-m,-n:1+b-c,c:1,1).$$

Now expanding $F_2(b : -m, -n : 1 + b - c, c : 1, 1)$ in series of p and q, replacing m and n by $m + p$ and $n + q$ respectively and on summing the series for m and n we get (3.1).

The result (3.2) can be proved on the parallel lines of (3.1). To prove (3.3) and (3.4) we make use of the following modified forms of (2.1) viz.,

$$F_2(a - m - n : -m, -n : 1 + a - e - n, e - m : 1, 1) = \frac{(e-a)_{2n-m}(1-e)_{2m-n}}{(1-e)_m(e-a)_n}(-)^{m+n} \quad \text{...(3.5)}$$

and

$$F_2(a - m : -m, -n : 1 + a - e - m, e : 1, 1) = \frac{(1-e)_{m-n}(e-a)_n}{(e-a)_m}(-)^{m+n} \quad \text{...(3.6)}$$

It may be mentioned that if we use the sum (2.1) in Horn's functions G_A and G_B of three variables (Pandey 1963), we get the following transformations :

$$G_A\left(\frac{1}{a}, a, a : b, b', b : \frac{1}{c}, c, c : x, y, z\right) = (1 - x - z)^{-b}(1-y)^{-b'}$$
$$\times F_F\left(c - a, c - a, c - a : b, b', b : 1 - a, c, c : \frac{-x}{1-x-z}, \frac{-y}{1-y}, \frac{-z}{1-x-z}\right). \quad \text{...(3.7)}$$

and

$$G_B\left(\frac{1}{a}, a, a' : b_1, b_2, b_3 : \frac{1}{c}, c, c : x, y, z\right) = (1-x)^{-b_1}(1-y)^{-b_2}(1-z)^{-b_3}$$
$$\times F_G\left(c - a, c - a, c - a : b_1, b_2, b_3 \times 1 - a, c, c: \frac{-x}{1-x}, \frac{-y}{1-y}, \frac{-z}{1-z}\right) \quad \text{...(3.8)}$$

where F_F and F_G are Saran's (1954) functions of three variables.

However, these transformations have already been proved by different method (Pandey 1963).

In the end it may be remarked that sinee

$$ {}_2F_1\left(\begin{matrix} m-n, a \\ e \end{matrix} ; 1\right) = \frac{(e-a)_{n-m}}{(e)_{n-m}}, \qquad m < n \quad \text{...(3.9)}$$

by Vandermonde's theorem we have from (2.1) and (3.9)

$$ {}_2F_1\left(\begin{matrix} m-n, a \\ e \end{matrix} ; 1\right) = F_2(a : -m, -n : 1 + a - e, e : 1, 1), \quad m < n.$$

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