METRIC SPACES OF ENTIRE FUNCTIONS*

by W. C. Sisarcick, *Department of Mathematics, Marshall University, Huntington, W. Va., U.S.A.*

(Communicated by S. M. Shah, F.N.A.)

(*Received* 12 *June* 1973)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be entire functions. Define $d(f, g) = \sup \{ | a_0 - b_0 |, | a_n - b_n |^{1/n}, n \geqslant 1\}$. If E denotes the class of entire functions of exponential type and B the class of functions of bounded index, then the metric spaces (E, d), (B, d), and $(E \setminus B, d)$ are all first category spaces that are not topologically complete.

If Γ denotes the class of entire functions, a characterization of the continuous linear functionals on (Γ, d) is obtained. Also the linear functionals on (Γ, d) satisfying a Lipschitz condition are characterized.

1. Introduction

Let Γ denote the class of entire functions. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n \in \Gamma,$$

a metric d is defined on Γ by

$$d(f, g) = \sup \{ | a_0 - b_0 |, | a_n - b_n |^{1/n}, n \geqslant 1\}. \qquad ...(1.1)$$

The metric space (Γ, d) has been studied extensively by Iyer (1948, 1956).

An entire function f is said to be of exponential type if and only if $\overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{r} < \infty$. It has recently been shown (Fricke *et al.* 1973) that an entire function f is of exponential type if and only if there exists a non-negative integer N such that for all $r > 0$ and all non-negative integers k,

$$\max_{0 \leqslant j \leqslant N} \frac{M(r, f^{(j)})}{j!} \geqslant \frac{M(r, f^{(k)})}{k!}. \qquad ...(1.2)$$

Let E denote the class of functions of exponential type. It is shown here that the metric space (E, d) is a first category space.

*The work done on this paper was supported by grant No. G.U. 3803.

An entire function f is said to be of bounded index if and only if there exists a non-negative integer N such that for all z and all non-negative integers k,

$$\max_{0 \leqslant j \leqslant N} \frac{|f^{(j)}(z)|}{j!} \geqslant \frac{|f^{(k)}(z)|}{k!}. \qquad \ldots(1.3)$$

It is known (Shah 1968, 1970) that a function of bounded index is of exponential type, but there exist functions of exponential type not of bounded index. Let B denote the class of functions of bounded index. It is known (Ekblaw 1971) that (B, d) is a first category space. This result is developed here in a somewhat different manner. It is also shown here that $(E \setminus B, d)$ is a first category space.

A metric space (X, ρ) is said to be topologically complete if and only if there is a metric ρ' defined on X inducing the same topology as ρ. Alexandroff shows (Oxtoby 1971) that if (X, ρ) is a complete metric space and y is a subset of X, then (y, ρ) is topologically complete if and only if y is a G_δ subset of X. It will be shown here that each of (E, d), (B, d) and $(E \setminus B, d)$ is not topologically complete.

A linear functional on Γ is a mapping F from Γ into $\mathbb{C}$ such that for all $f, g \in \Gamma$, $\alpha, \beta \in \mathbb{C}$, $F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$. A continuous linear functional on Γ will be a linear functional on Γ continuous with respect to the topology generated by d and the usual topology on $\mathbb{C}$. It is known (Iyer 1948) that a linear functional F on Γ is continuous if and only if there exists a unique sequence $\{b_n\}_{n=0}^{\infty}$ in $\mathbb{C}$ such that

$\{|b_n|^{1/n}\}_{n=1}^{\infty}$ is bounded and for all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$,

$$F(f) = \sum_{n=1}^{\infty} a_n b_n. \qquad \ldots(1.4)$$

A characterization of continuous linear functionals in terms of $M(r, f)$ will be given here.

Let F be a linear functional on Γ. If there exist two positive constants M and α such that for all $f, g \in \Gamma$,

$$|F(f) - F(g)| \leqslant M\,[d(f, g)]^{\alpha} \qquad \ldots(1.5)$$

then F is said to satisfy a Lipschitz condition of order α.

The linear functionals on Γ satisfying a Lipschitz condition of order α, α any positive real number, are characterized here.

2. The Metric Spaces (E, d), (B, d), $(E \setminus B, d)$

Theorem 1 — The metric space (E, d) is a first category space.

To prove Theorem 1, two lemmas are requircd. For each non-negative integer n, let

$$E_n = \left\{ f \in \Gamma \mid \text{for all } r > 0, \text{ all } k, \max_{0 \leqslant j \leqslant n} \frac{M(r, f^{(j)})}{j!} \geqslant \frac{M(r, f^{(k)})}{k!} \right\}.$$

Lemma 1 — Each E_n is a closed subset of Γ. *

PROOF : For a fixed n, let $f \in \Gamma$ be a limit point of E_n and $\{f_m\}_{m=1}^{\infty}$ a sequence in E_n converging to f. Since convergence with respect to d is equivalent to uniform convergence on compact subsets of $\mathbb{C}$ (Iyer 1948) for each non-negative integer k, $\{f_m^{(k)}\}_{m=1}^{\infty}$ converges uniformly on compact subsets of $\mathbb{C}$ to $f^{(k)}$, and hence converges to $f^{(k)}$ with respect to d.

Let $\epsilon > 0$, $r > 0$, $k \geqslant 0$ be given. Then there exists an N such that

$$| f_N^{(j)}(t) - f^{(j)}(t) | < \frac{\epsilon j!}{2}$$

if $| z | = r$ and $j \in \{0, 1, 2 \ldots n, k\}$. From this one easily obtains

$$\frac{M(r, f^{(k)})}{k!} < \frac{\epsilon}{2} + \frac{M(r, f_N^{(k)})}{k!}$$

and for $j \in \{0, 1 \ldots n\}$,

$$\frac{M(r, f_N^{(j)})}{j!} < \frac{\epsilon}{2} + \frac{M(r, f^{(j)})}{j!}.$$

Since $f_N \in E_n$, $\dfrac{M(r, f^{(k)})}{k!} < \dfrac{\epsilon}{2} + \dfrac{M(r, f_N^{(k)})}{k!}$

$$\leqslant \frac{\epsilon}{2} + \max_{0 \leqslant j \leqslant n} \frac{M(r, f_N^{(j)})}{j!} \leqslant \epsilon + \max_{0 \leqslant j \leqslant n} \frac{M(r, f^{(j)})}{j!}.$$

Let $\epsilon \to 0$ and get $f \in E_n$. Thus, E_n is a closed subset of Γ.

Lemma 2 — Each $E \setminus E_n$ is a dense subset of Γ.

PROOF : Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$ and $\epsilon > 0$ be given. Let $h(z) = \sum_{n=0}^{\infty} b_n z^n \in \Gamma \setminus E$.

*This and all subsequent statements of a topological nature mean with respect to the topology generated by d on the set in question.

Choose m so that $h_m(z) = \sum_{n=0}^{m} (a_n - b_n) z^n + h(z)$ satisfies $d(h_m, f) < \epsilon/2$. Since E is closed under addition, $h_m \in \Gamma \setminus E$. By Lemma 1, there exists a δ, $0 < \delta < \epsilon/2$ such that $g \in \Gamma$ and $d(g, h_m) < \delta$ gives $g \in \Gamma \setminus E_n$. Since the polynomials in Γ are dense in Γ, choose polynomial p such that $d(p, h_m) < \delta$. Then $p \in E \setminus E_n$ and $d(p, f) < \epsilon$. Thus E/E_n is dense in Γ.

PROOF OF THEOREM 2.1 — Since an entire function f is of exponential type if and only if f satisfies (1.2), $E = \bigcup_{n=0}^{\infty} E_n$. By Lemma 1, each E_n is a closed subset of E. It remains to show $int_E E_n = \phi$. Applying lemma 2, $int_E E_n = E \setminus \overline{E \setminus E_n}^E = E \setminus (\overline{E \setminus E_n}^\Gamma \cap E) = E \setminus (\Gamma \cap E) = E \setminus F = \phi$.

Theorem 2.2 — The metric space (B, d) is a first category space.

PROOF : As in the proof of Theorem 1, $E = \bigcup_{n=0}^{\infty} E_n$. Since $B \subseteq E$ (Shah 1968), $B = E \cap B = \bigcup_{n=0}^{\infty} (B \cap E_n)$. Since each E_n is closed in E, each $B \cap E_n$ is closed in B. It remains to show $int_B\, B \cap E_n = \phi$. Since each polynomial is a function of bounded index, the proof of Lemma 2 gives the stronger result that each $B \setminus E_n$ is a dense subset of Γ. Thus $int_B\, B \cap E_n = B \setminus \overline{B \setminus (B \cap E_n)}^B = B \setminus (\overline{B \setminus (B \cap E_n)}^\Gamma \cap B)$ $= B \setminus (\overline{B \setminus E_n}^\Gamma \cap B) = B \setminus (\Gamma \cap B) = B \setminus B = \phi$.

Theorem 2.3 — The metrie space $(E \setminus B, d)$ is a first category space.

The proof of this theorem will require the use of Lemma 2 and an additional lemma.

Lemma 3 — The set $E \setminus B$ is a dense subset of E.

PROOF : Let $f \in E$ and $\epsilon > 0$ be given. We show there exists an $h \in E \setminus B$ such that $d(h, f) < \epsilon$. We may assume $f \not\equiv 0$ since there are functions not identically zero on $\mathbb{C}$ that are in E and are arbitrarily close to the function that is identically zero on $\mathbb{C}$.

Choose $R > \frac{2}{\epsilon}$ and then choose $a > 1$ so that $\frac{a}{(a-1)^2} \leqslant \frac{1}{R} \ln \left\{ \frac{M(R, f) + 1}{M(R, f)} \right\}$.

Let $g(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a^k}\right)^k$. Then $g \in E \setminus B$ and it follows easily from Laguerre's theorem (Boas 1954) that $| g^{(k)}(0) | \leqslant | g'(0) |^k$ for each positive integer k. Thus

$$1-g(z) = -\sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} z^k \text{ and } M(R, 1-g) \leqslant \sum_{k=1}^{\infty} \frac{|g^{(k)}(0)|}{k!} R^k$$

$$\leqslant \sum_{k=1}^{\infty} \frac{|g'(0)|^k}{k!} R^k = e^{|g'(0)|R} - 1 \leqslant \exp\left\{\frac{aR}{(a-1)^2}\right\} - 1 \leqslant \frac{M(R,f)+1}{M(R,f)} - 1$$

$$= \frac{1}{M(R,f)}.$$ Hence $M(R,f) \,.\, M(R, 1-g) \leqslant 1$.

Let $h(z) = f(z) \,.\, g(z)$. Then $h \in E \setminus B$ and for $k \geqslant 1$, $\left|\frac{f^{(k)}(0)}{k!} - \frac{h^{(k)}(0)}{k!}\right| \leqslant \frac{M(R,f) \,.\, M(R, 1-g)}{R^k} \leqslant \frac{1}{R^k}$ and so $\left|\frac{f^{(k)}(0)}{k!} - \frac{h^{(k)}(0)}{k!}\right|^{1/k} \leqslant \frac{1}{R} < \frac{\epsilon}{2}$. Since $h(0) = f(0)$, $d(h, f) \leqslant \frac{\epsilon}{2} < \epsilon$.

Proof of Theorem 2.3 — Since $E = \bigcup_{n=0}^{\infty} E_n$ and each E_n is closed in E, $E \setminus B = \bigcup_{n=0}^{\infty} E_n \setminus B$ and each $E_n \setminus B$ is closed in $E \setminus B$. It remains to show $int_{E \setminus B}\, E_n \setminus B = \phi$ for each n. We first show $E \setminus (B \cup E_n)$ is dense in E for each n. Let $f \in E$ and $\epsilon > 0$. By Lemma 2, there is an $h \in E \setminus E_n$ such that $d(h, f) < \epsilon/2$. By Lemma 1, there is a δ, $0 < \delta < \epsilon/2$ such that if $g \in \Gamma$ and $d(g, h) < \delta$, then $g \notin E_n$. By Lemma 3, there is a $g \in E \setminus B$ such that $d(g, h) < \delta$. Hence, $g \in E \setminus (B \cup E_n)$ and $d(g, f) < \epsilon$.

For each n, $int_{E \setminus B}\, E_n \setminus B$

$$= (E \setminus B) \setminus \overline{(E \setminus B) \setminus (E_n \setminus B)}^{E \setminus B}$$
$$= (E \setminus B) \setminus [\overline{(E \setminus B) \setminus (E_n \setminus B)}^{E} \cap (E \setminus B)]$$
$$= (E \setminus B) \setminus [\overline{E \setminus (B \cup E_n)}^{E} \cap (E \setminus B)]$$
$$= (E \setminus B) \setminus [E \cap (E \setminus B)] = (E \setminus B) \setminus (E \setminus B) = \phi.$$

Theorem 2.4 — Each of (E, d), (B, d) and $(E \setminus B, d)$ is not topologically complete.

Proof: Since (Γ, d) is complete (Iyer 1948), by Alexandroff's Theorem, if sufficies to show none of E, B or $E \setminus B$ is a G_δ subset of Γ.

Assume $E = \bigcap_{j=1}^{\infty} A_j$, each A_j an open subset of Γ. Then $\Gamma \setminus E = \bigcup_{j=1}^{\infty} \Gamma \setminus A_j$ and each $\Gamma \setminus A_j$ is a closed subset of Γ. Since each A_j contains all the polynomials, $int_\Gamma \Gamma \setminus A_j = \phi$. Thus $\Gamma \setminus E$ is a first category subset of Γ. Since E is a first category subset of Γ, (Γ, d) is a first category space. But this contradicts the Baire Category Theorem. Hence E, and similarly B, is not a G_δ subset of Γ.

Assume $E \setminus B = \bigcap_{j=1}^{\infty} C_j$, each C_j an open subset of Γ. Then $(\Gamma \setminus E) \cup B = \bigcup_{j=1}^{\infty} \Gamma \setminus C_j$ and each $\Gamma \setminus C_j$ is a closed subset of Γ. Since $\bar{E}^{\Gamma} = \Gamma$, given $f \in \Gamma \setminus C_j$ and $\epsilon > 0$, there is a $g \in E$ such that $d(f, g) < \epsilon/2$. By Lemma 3, there is an $h \in E \setminus B$ such that $d(h, g) < \epsilon/2$. Hence, $g \in E \setminus B$ with $d(g, f) < \epsilon$ and $int_{\Gamma} \Gamma \setminus C_j = \phi$. Since $(E \setminus B, d)$ is a first category space and (Γ, d) is a complete space, the Baire Category Theorem is again contradicted.

3. Linear Functionals on Γ

Theorem 3.1 — Let F be a linear functional on Γ. Then F is continuous if and only if there exists a $K \geqslant 0$ such that for all $f \in \Gamma$,

$$| F(f) | \leqslant KM(K+1, f) \qquad \ldots(3.1)$$

Proof: Let F be continuous. Then F satisfies (1.4). Let $K = 2 \sup \{ | b_0 |, | b_n |^{1/n}, n \geqslant 1\}$. Then $| F(f) | \leqslant \sum_{n=0}^{\infty} | a_n | \, | b_n | \leqslant \sum_{n=0}^{\infty} \frac{M\left(\frac{K}{2}+1, f\right)}{\left(\frac{K}{2}+1\right)^n} | b_n |$

$$\leqslant M(K+1, f) \left\{ | b_0 | + \sum_{n=1}^{\infty} \frac{| b_n |}{\left(\frac{K}{2}+1\right)^n} \right\}$$

$$\leqslant M(K+1, f) \left\{ \frac{K}{2} + \sum_{n=1}^{\infty} \frac{\left(\frac{K}{2}\right)^n}{\left(\frac{K}{2}+1\right)^n} \right\} = KM(K+1, f).$$

Conversely, let F be a linear functional on Γ satisfying (3.1). Let $f_0 \in \Gamma$ be such that $f_0(z) = 0$ for all $z \in ¢$. It is sufficient to show F is continuous at f_0. Given $\epsilon > 0$, choose $\delta > 0$ such that $K \left\{ \delta + \frac{\delta(K+1)}{1-\delta(K+1)} \right\} < \epsilon$ and $\delta(K+1) < 1$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$ with $d(f, f_0) < \delta$, then $KM(K+1, f) \leqslant K \sum_{n=0}^{\infty} | a_n | (K+1)^n$

$$\leqslant K \left\{ \delta + \sum_{n=1}^{\infty} \delta^n (K+1)^n \right\} = K \left\{ \delta + \frac{\delta(K+1)}{1-\delta(K+1)} \right\} < \epsilon.$$

Since F is continuous, F satisfies (1.4) and so

$$\left| \sum_{n=0}^{\infty} a_n b_n \right| \leqslant KM(K+1, f). \qquad \ldots(3.2)$$

We now show $\sup \{ | b_0 | , | b_n |^{1/n}\} \leqslant \max \{2, K(K+1)\}$. Choosing $f(z) \equiv 1$, from (3.2) it follows that $| b_0 | \leqslant K \leqslant \max \{2, K(K+1)\}$. Choosing $f(z) = z^n, n \geqslant 1$, (3.2) becomes $| b_n |^{1/n} \leqslant K^{1/n} (K+1) \leqslant \max \{2, K(K+1)\}$.

Let F be a linear functional on Γ. By the linearity of F, F satisfies a Lipschitz condition of order α if and only if there exist two positive constants M and α such that for all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$,

$$| F(f) | \leqslant M [d(f, f_0)]^{\alpha}. \qquad ...(3.3)$$

Here $f_0(z) = 0$ for all $z \in ¢$.

A function satisfying a Lipschitz condition must be continuous, hence $F(f) = \sum_{n=0}^{\infty} a_n b_n$ and (3.3) may be written as

$$\left| \sum_{n=0}^{\infty} a_n b_n \right| \leqslant M \{\sup \{ | a_0 | , | a_n |^{1/n}, n \geqslant 1\}\}^{\alpha}. \qquad ...(3.4)$$

Theorem 3.2 — Let F be a linear functional on Γ satisfying a Lipschitz condition of order α. Then the following hold.

(1) If α is not an integer, $F \equiv 0$.

(2) If α is an integer and $\alpha \geqslant 2$, there exists a unique $b \in ¢$ such that for all $f \in \Gamma$,

$$F(f) = \frac{f^{(\alpha)}(0)}{\alpha!} b.$$

(3) If $\alpha = 1$, there exist unique complex numbers a and b such that for all $f \in \Gamma$, $F(f) = af(0) + bf'(0)$.

PROOF OF (1) : Choose $a_0 \neq 0$ and $a_n = 0$ if $n \geqslant 1$. By (3.4), $| a_0 b_0 | \leqslant M | a_0 |^{\alpha}$ or $| b_0 | \leqslant M | a_0 |^{\alpha-1}$. If $\alpha < 1$ let $a_0 \to \infty$ to get $b_0 = 0$. If $\alpha > 1$ let $a_0 \to 0$ to get $b_0 = 0$. For fixed positive integer k, let $a_k \neq 0$ and $a_n = 0$ if $n \neq k$. Then $| a_k b_k | \leqslant M | a_k |^{\alpha/k}$ or $| b_k | \leqslant M | a_k |^{(\alpha/k)-1}$. If $\alpha < k$ let $| a_k | \to \infty$ to get $b_k = 0$. If $\alpha > k$ let $a_k \to 0$ to get $b_k = 0$. Therefore each $b_k = 0$ and $F \equiv 0$.

PROOF OF (2) : As in the proof of (1), $b_k = 0$ if $k \neq \alpha$ and $F(f) = a_\alpha b_\alpha = \frac{f^{(\alpha)}(0)}{\alpha!} b_\alpha$. For $f(z) = a_\alpha z^\alpha \in \Gamma$, $| F(f) | = | a_\alpha b_\alpha | = | b_\alpha | [d(f, f_0)]^{\alpha} \leqslant M[d(f, f_0)]^{\alpha}$, thus $| b_\alpha | \leqslant M$. That this cannot be improved upon is seen by the fact that if b is any complex number and if $G : \Gamma \to ¢$ is defined by $G(f) = \frac{f^{(\alpha)}(0)}{\alpha!} b$, then G is a continuous linear functional satisfying a Lipschitz condition of order α with $| b | \leqslant M$.

PROOF OF (3) : As in the proof of (1), $b_k = 0$ if $k \geqslant 2$ and $F(f) = a_0 b_0 + a_1 b_1 = b_0 f(0) + b_1 f'(0)$. Conversely, given any two complex numbers a and b, define $F : \Gamma \to ₵$ by $F(f) = af(0) + bf'(0)$. Then F is a continuous linear functional satisfying a Lipschitz condition of order 1.

Multiplication may be defined on Γ naturally in three ways. Given $f, g \in \Gamma$, define $(f \cdot g)(z) = f(z) \cdot g(z)$, $z \in ₵$. If F is a $\not\equiv 0$ continuous linear functional on Γ, it is known (Sisarcick 1973) that $F(f \cdot g) = F(f) \cdot F(g)$ for all $f, g \in \Gamma$ if and only if there exists a unique $b \in ₵$ such that for all $f \in \Gamma$, $F(f) = f(b)$.

Secondly we may define $f \cdot g$ by $(f \cdot g)(z) = f(g(z))$, $z \in ₵$. Using this definition of multiplication, we have the following Theorem.

Theorem 3.3 — A function $F : \Gamma \to ₵$ is a continuous linear functional satisfying $F(f \cdot g) = F(f) \cdot F(g)$ for all $f, g \in \Gamma$ if and only if $F \equiv 0$.

PROOF : Let such an F be given. Then there exists $\{c_n\}_{n=0}^{\infty}$ such that for all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$, $F(f) = \sum_{n=0}^{\infty} a_n c_n$. Let $f(z) \equiv b$, b arbitrary, $b \in ₵$. Then $bc_0 = F(f) = F(f \cdot f) = (F(f))^2 = (bc_0)^2$. Since b is arbitrary, $c_0 = 0$. Let $f(z) = bz^n$, b arbitrary, $b \in ₵$, n fixed, $n \geqslant 2$. Then $(f \cdot f)(z) = b^{n+1} z^{n^2}$, $(bc_n)^2 = (F(f))^2 = F(f \cdot f) = b^{n+1} c_{n^2}$. Since b is arbitrary, $c_n = 0$. Let $f(z) = z$. Then $(c_1)^2 = (F(f))^2 = F(f \cdot f) = c_1$, and so $c_1 = 1$ or $c_1 = 0$. If $c_1 = 1$, $F(f) = f'(0)$ for all $f \in \Gamma$. Taking $f(z) = \sin z$, $g(z) = e^z$, it follows that $F(f \cdot g) \neq F(f) \cdot F(g)$. Thus $c_1 = 0$ and hence $F \equiv 0$.

Thirdly, we may define multiplication on Γ by using Hadamard composition. That is, if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \Gamma$, define $(f \circ g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Theorem 3.4 — Let $F \not\equiv 0$ be a continuous linear functional on Γ. Then $F(f \circ g) = F(f) \cdot F(g)$ for all $f, g \in \Gamma$ if and only if

$$F(f) = \frac{f^{(j)}(0)}{j!} \text{ for some fixed } j.$$

PROOF : Let F be a $\not\equiv 0$ continuous linear functional on Γ. Then there exists $\{c_n\}_{n=0}^{\infty}$ such that for all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$, $F(f) = \sum_{n=0}^{\infty} a_n c_n$. Let $F(f \circ g) = F(f) \cdot F(g)$ for all $f, g \in \Gamma$. For fixed non-negative integer n, let $f(z) = z^n$. Then $c_n^2 = (F(f))^2 = F(f \circ f) = F(f) = c_n$. Thus $c_n = 0$ or 1. Assume $c_m = 1 = c_n$,

$m \neq n$, m and n non-negative integers. Let $g(z) = z^m + z^n$. Then $4 = (c_m + c_n)^2 = (F(g))^2 = F(g \circ g) = F(g) = c_m + c_n = 2$. Thus at most one c_j is non-zero. Since $F \not\equiv 0$, there is exactly one $c_j \neq 0$. Thus $F(f) = \dfrac{f^{(j)}(0)}{j!}$.

Conversely, for fixed j, define $F : \Gamma \to \mathbb{C}$ by $F(f) = \dfrac{f^{(j)}(0)}{j!}$. Then F is clearly a $\not\equiv 0$ continuous linear functional satisfying $F(f \circ g) = F(f) \cdot F(g)$ for all $f, g \in \Gamma$.

References

Boas, R. P. (1954). Entire Functions. Academic Press, New York.

Ekblaw, K. A. (1971). The functions of bounded index as a subspace of a space of entire functions. *Pacific J. Math.*, **37**, 353-55.

Fricke, G. H., Shah, S. M., and Sisarcick, W. C. (1973). A characterization of entire functions of exponential type and *M*-bounded index. *Indiana Univ. Math. J.*, (to appear).

Iyer, V. G. (1948). On the space of integral functions—I. *J. Indian math. Soc.*, **12**, 13-30.

———— (1956). On the space of integral functions—IV. *Proc. Am. math. Soc.*, **7**, 644-49.

Oxtoby, John C. (1971). Measure and Category. Springer-Verlag, Berlin.

Shah, S. M. (1968). Entire functions of bounded index. *Proc. Am. math. Soc.*, **19**, 1017-22.

———— (1970). Entire functions of unbounded index and having simple zeros. *Math. Z.*, **118**, 193-96.

Sisarcick, W. C. (1973). Linear functionals on the space of entire functions. *Proc. W. Va. Acad. Sci.*, (to appear).