RESULTS INVOLVING GENERALIZED HYPERGEOMETRIC POLYNOMIALS AND ORTHOGONAL POLYNOMIALS RELATED TO HERMITE POLYNOMIALS

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This is in continuation of the study to the polynomial solutions of the following self-adjoint differential equations :

$$D\left[e^{-x^{2k}} Dy\right] + \lambda x^{2k-2} e^{-x^{2k}} y = 0$$

$$D \equiv \frac{d}{dx}; \; k = 1, 2, 3, \ldots .$$

The above polynomial solutions lead us to even $H_{2pk}(x; k)$ and odd $H_{2pk+1}(x; k)$, $p = 0, 1, 2, \ldots$ orthogonal polynomials.
Some integrals involving these polynomials and the generalized hypergeometric polynomials introduced by Shah are evaluated. Expansions of some functions in terms of the orthogonal polynomials $H_{2pk}(x; k)$ (or $H_{2pk+1}(x; k)$) are also obtained and particular cases of the results are enlisted.

1. INTRODUCTION

Thakare and Karande (1973) studied the class of orthogonal polynomials which are related to Hermite polynomials. They discussed the polynomial solutions of the following self-adjoint differential equations.

$$D\left[e^{-x^{2k}} Dy\right] + \lambda x^{2k-2} e^{-x^{2k}} y = 0 \qquad \ldots(1.1)$$

$$D \equiv \frac{d}{dx}; \; k = 1, 2, 3, \ldots . \qquad \ldots(1.2)$$

In fact the polynomial solutions lead us to the even $H_{2pk}(x; k)$ and odd $H_{2pk+1}(x; k)$ polynomials according to the even and odd values of λ as given by

$$\left.\begin{array}{l} \lambda_p^E = 2pk \\ \lambda_p^O = 2pk + 1 \end{array}\right\} \begin{array}{l} k = 1, 2, 3, \ldots, \\ p = 0, 1, 2, \ldots . \end{array}$$

These polynomials are orthogonal w.r.t. the weight function $x^{2k-2} \exp(-x^{2k})$ over the interval $(-\infty, \infty)$.

Among other results we have established is the following Rodrigues' formula

$$H_{2pk}(x; k) = H_{2pk}(z) = \frac{(-1)^p (p+1)_p}{(1+\beta)_p} z^{-\beta} e^z \frac{d^p}{dz^p} [e^{-z} z^{p+\beta}] \quad \ldots(1.3)$$

for the even polynomials and

$$H_{2pk+1}(x; k) = H_{2pk+1}(z) = \frac{(-1)^p (p+1)_p}{(1+\beta)_p} z^{-\beta} e^z \frac{d^p}{dz^p} [e^{-z} z^{p+\beta}] \quad \ldots(1.4)$$

for the odd polynomials. In both the formulae we have $z = x^{2k}$, $-\beta = 1/2k$. It is remarked that for $k = 1$ the solutions of (1.1) constitute the Hermite polynomials.

Shah (1967) defined the generalised hypergeometric polynomials by means of

$$F_m(x) = x^{(\delta-1)m} {}_{p+\delta}F_q \left[\begin{matrix} \triangle(\delta, -m), a_1, a_2, \ldots, a_p ; \\ b_1, b_2, \ldots, b_q ; \end{matrix} \mu x^c \right] \quad \ldots(1.5)$$

where m and δ are positive integers and the symbol $\triangle(\delta, -m)$ denotes the set of δ-parameters :

$$\frac{-m}{\delta}, \frac{-m+1}{\delta}, \ldots \frac{-m+\delta-1}{\delta}.$$

The purpose of this paper is to evaluate some integrals involving the polynomials $H_{2pk}(x; k)$ (or $H_{2pk+1}(x ; k)$) and the generalized hypergeometric polynomials $F_m(x)$. As application we shall obtain some expansions of some functions (involving generalized hypergeometric polynomials) in terms of the orthogonal polynomials $H_{2pk}(x; k)$ (or $H_{2pk+1}(x; k)$).

2. Integrals

In the course of our investigation we shall need the following integral :

$$I_1 = \int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k}) x^{2pk+2sk} H_{2pk}(x ; k) \, dx.$$

On account of the Rodrigues' formula (1.3) we obtain

$$I_1 = \frac{(-1)^{p+1} (1+p)_p 2\beta}{(1+\beta)_p} \int_0^{\infty} z^{p+s} \frac{d^p}{dz^p} \{e^{-z} z^{p+\beta}\} \, dz.$$

Integrating by parts p times we have

$$I_1 = \frac{-2\beta(1+p)_p(1+s)_p}{(1+\beta)_p}\,\Gamma(p+\beta+s+1), \qquad \ldots(2.1)$$

provided that $\operatorname{Re}(p+\beta+s+1) > 0$.

We shall demonstrate and exhibit the use of the integral (2.1) in evaluating the more general integral

$$I_2 = \int_{-\infty}^{\infty}\Big\{x^{2k-2}\exp(-x^{2k})\,x^{2pk+2ps}\,H_{2pk}(x;k)$$

$$\times\, x^{(\delta-1)m}\;{}_{p'+\delta}F_q\left[\begin{matrix}\Delta(\delta,-m),(a_{p'})\,; \\ (b_q);\end{matrix}\;\lambda x^{-2ck}\right]\Big\}\,dx.$$

This integral I_2 can also be written as

$$I_2 = \sum_r \frac{\prod_{i=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r (a_{p'})_r\,\lambda^r}{(b_q)_r\, r\,!}\times$$

$$\times\int_{-\infty}^{\infty} x^{2k-2}\exp(-x^{2k})\,x^{2pk+2sk+(\delta-1)m-2crk}\,H_{2pk}(x;k)\,dx.$$

It is to be noted that the summation above is finite.

$$I_2 = \sum_r \frac{\prod_{j=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r (a_{p'})_r\,\lambda^r(-2\beta)(1+p)_p}{(b_q)_r\, r!\,(1+\beta)_p}$$

$$\times\frac{\Gamma\left(p+s+\frac{(\delta-1)m}{k}-cr+1\right)\Gamma\left(p+s+\frac{(\delta-1)m}{2k}+\beta+1-cr\right)}{\Gamma\left(s+\frac{(\delta-1)m}{k}-cr+1\right)}$$

$$= \frac{(-2\beta)(1+p)_p}{(1+\beta)_p}\;\frac{\Gamma\left(1+p+s+\left(\frac{\delta-1}{2k}\right)m\right)}{\Gamma\left(1+\delta+\left(\frac{\delta-1}{2k}\right)m\right)}\times$$

(*equation continued on p. 640*)

$$\times \Gamma\left(1+p+s+\beta+\frac{(\delta-1)m}{2k}\right)$$

$$\times \sum_r \left[\frac{\prod_{j=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r (a_{p'})_r \lambda^r (-1)^{cr}\left(-s-\frac{(\delta-1)m}{2k}\right)_{cr}}{(b_q)_r\, r!\left(-p-s-\frac{(\delta-1)m}{2k}\right)_{rc}\left(-p-s-\beta-\frac{(\delta-1)m}{2k}\right)_{cr}}\right]$$

$$= \frac{(-2\beta)(1+p)_p\,\Gamma\left(1+p+\delta+\frac{(\delta-1)m}{2k}\right)\Gamma\left(1+p+s+\beta+\frac{\delta-1}{2k}m\right)}{(1+\beta)_p\,\Gamma\left(1+s+\frac{\delta-1}{2k}m\right)}$$

$$\times \sum_r \left\{\frac{\prod_{i=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r (a_{p'})_r \lambda^r(-1)^{cr}}{(b_q)_r\, r!\, c^{cr}\prod_{j=0}^{c-1}\left(\frac{\frac{-2pk-2sk-(\delta-1)m}{2k}+j}{c}\right)_r}\right.$$

$$\left.\times \frac{\prod_{i=0}^{c-1}\left(\frac{\frac{-2\delta k-(\delta-1)m}{2k}+i}{c}\right)_r}{\prod_{j=0}^{c-1}\left(\frac{1-2pk-2sk-(\delta-1)m}{2k}+j\right)_r}\right\}.$$

Hence, we can put this in the form

$$\int_{-\infty}^{\infty} x^{2k-2}\exp(-x^{2k})\, x^{2pk+2sk+(\delta-1)m} H_{2pk}(x;k)$$

$$\times\ {}_{p'+\delta}F_q\left[\begin{matrix}\Delta(\delta,-m),(a_{p'});\\ (b_q);\end{matrix}\ \lambda x^{-2ck}\right]dx$$

$$= \frac{(-2\beta)(1+p)_p\,\Gamma\left(1+p+s+\frac{(\delta-1)m}{2k}\right)\Gamma\left(1+p+s+p+\frac{(\delta-1)m}{2k}\right)}{\Gamma\left(1+s+\frac{(\delta-1)m}{2k}\right)(1+\beta)_p}\times$$

(*equation continued on p. 641*)

$$\times {}_{p'+\delta+c}F_{q+2c}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta\left(c, \frac{-2sk-(\delta-1)m}{2k}\right),\ (a_{p'});\\ \qquad\qquad\qquad\qquad\qquad\qquad \frac{\lambda(-1)^c}{c^c}\\ (b_q),\ \Delta\left(c, \frac{-2pk-2sk-(\delta-1)m}{2k}\right),\ \Delta\left(c, \frac{1-2pk-2sk-m(\delta-1)}{2k}\right);\end{matrix}\right]. \qquad \ldots(2.2)$$

In a like manner, we can obtain the following integral involving the odd polynomials $H_{2pk+1}(x; k)$

$$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k})\, x^{2pk+2sk+(\delta-1)m+1} H_{2pk+1}(x; k)$$

$$\times {}_{p'+\delta}F_q\left[\begin{matrix}\Delta(\delta,-m),\ (a_{p'});\\ \qquad\qquad \lambda x^{-2ck}\\ (b_q);\end{matrix}\right] dx$$

$$= \frac{(-2\beta)\, 2^{2p+1} (\tfrac{3}{2})_p\, \Gamma\left(1+p+s+\frac{(\delta-1)m}{2k}\right) \Gamma\left(1+p+s-\beta+\frac{(\delta-1)m}{2k}\right)}{(1-\beta)_p\, \Gamma\left(1+\delta+\frac{(\delta-1)m}{2k}\right)}$$

$$\times {}_{p'+\delta+c}F_{q+2c}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta\left(c, \frac{-2sk-(\delta-1)m}{2k}\right),\ (a_{p'});\\ \qquad\qquad\qquad\qquad\qquad\qquad \frac{(-1)^c\lambda}{c^c}\\ (b_q),\ \Delta\left(c, \frac{-2pk-2sk-(\delta-1)m}{2k}\right),\ \Delta\left(c, \frac{-2pk-2sk-1-(\delta-1)m}{2k}\right);\end{matrix}\right].$$

If we put $k=1$, we shall get respectively the results for even Hermite polynomials and odd Hermite polynomials.

The combination of these results will give the following result of Shah (1968) :

$$\int_{-\infty}^{\infty} e^{-x^2} x^{n+2s+(\delta-1)m} H_n(x)\, {}_{p'+\delta}F_q\left[\begin{matrix}\Delta(\delta,-m),\ (a_{p'});\\ \qquad\qquad \lambda x^{-2c}\\ (b_q);\end{matrix}\right] dx$$

$$= \frac{\Gamma(n+2s+(\delta-1)m+1)\sqrt{\pi}\, 2^{-2s-(\delta-1)m}}{\Gamma\left(\frac{2s+(\delta-1)m+2}{2}\right)} \times$$

(*equation continued on p. 642*)

$$\times {}_{p'+\delta+c}F_{q+2c}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta\left(c,\ \dfrac{-2s-(\delta-1)m}{2}\right),\ (a_p);\\ \\ \Delta(2c,\ -n-2s-(\delta-1)m),\ (b_q)\ ;\end{matrix}\ \ \frac{(-1)^c\,\lambda}{c^c}\right]$$

where c is a positive integer. The use of the following relation is also made

$$(\alpha)_{ns} = s^{ns}\prod_{j=1}^{s}\left(\frac{\alpha+1-j}{s}\right)_n.$$

3. Particular Cases

Put $\delta=\lambda=c=1$ in (2.2). Take also $a_1=m+\alpha+\beta+1$, $b_1=1+\alpha$, $b_2=\frac{1}{2}$. Multiplying both the sides by $\dfrac{(1+\alpha)_m}{m!}$, we have

$$\int_{-\infty}^{\infty} x^{2k-2}\exp(-x^{2k})\,x^{2pk+2sk}\,H_{2pk}(x\ ;k)$$

$$\times f_m^{(\alpha,\beta)}\left(\begin{matrix}a_2,\ldots,a_{p'}\ ;\\ \\ b_3,\ldots,b_q\ ;\end{matrix}\ \ x^{-2k}\right)dx$$

$$=\frac{(-2\beta)\,(1+p)_p\,\Gamma(1+p+s)\,\Gamma(1+p+s+\beta)}{(1+\beta)_p\,\Gamma(1+s)}$$

$$\times f_m^{(\alpha,\beta)}\left(\begin{matrix}a_2,\ldots,a_{p'},\,-s\ ;\\ \\ b_3,\ldots,b_q,\,-p-s,\,-\beta-s-p\ ;\end{matrix}\ \ -1\right) \qquad \ldots(3.1)$$

where

$$f_m^{(\alpha,\beta)}\left(\begin{matrix}a_2,\ldots,a_p\ ;\\ \\ b_3,\ldots,b_q\ ;\end{matrix}\ \ x\right)$$

$$=\frac{(1+\alpha)_m}{m!}\ {}_{p+1}F_q\left[\begin{matrix}-m,\ m+\alpha+\beta+1,\ a_2,\ldots,a_p\ ;\\ \\ 1+\alpha,\ \tfrac{1}{2},\ b_3,\ldots,b_q\ ;\end{matrix}\ \ x\right] \qquad \ldots(3.2)$$

is a generalized Sister Celine polynomial (Shah 1969) which reduces to Sister Celine's polynomials (Fasenmyer 1947) after putting $\alpha=\beta=0$.

If in (3.1) we put $k=1$, then we shall get a corresponding result involving even Hermite polynomials.

In a like manner, if we put $p' = q = 3$, $a_2 = \rho$, $a_3 = \frac{1}{2}$, $b_3 = \sigma$, we get

$$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k})\, x^{2pk+2sk}\, H_{2pk}(x;k)\, f_m^{(\alpha,\beta)}(\rho,\sigma;x^{-2k})\,dx$$

$$= \frac{(-2\beta)(1+p)_p\,\Gamma(1+p+s)\,\Gamma(1+p+s+\beta)(1+\alpha)_m}{(1+\beta)_p\,\Gamma(1+s)\,m!}$$

$$\times\ {}_4F_4\left[\begin{matrix} -m, m+\alpha+\beta+1, \rho, -s\,; \\ 1+\alpha, \sigma, -p-s, -\beta-s-p\,; \end{matrix}\ -1\right]$$

where

$$f_m^{(\alpha,\beta)}(\rho,\sigma\,;x) = \frac{(1+\alpha)_m}{m!}\ {}_3F_2\left[\begin{matrix} -m, m+\alpha+\beta+1, \rho\,; \\ 1+\alpha, \sigma\,; \end{matrix}\ x\right] \qquad \ldots(3.3)$$

is a generalized Rice's polynomial (Khandekar 1964) which reduces to Rice's polynomial when $\alpha = \beta = 0$.

Again if we put $k = 1$, we shall get a result involving even Hermite polynomials.

Take $p' = 0$, $q = 1$, $b_1 = 1 + \alpha$ in (2.2). Multiply both the sides by $\frac{(1+\alpha)_m}{m!}$ to get

$$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k})\, H_{2pk}(x\,;k)\, L_m^{\alpha}(x)\,dx$$

$$= \frac{(-2\beta)(1+p)_p\,\Gamma(1+p+s)\,\Gamma(1+p+s+\beta)(1+\alpha)_m}{(1+\beta)_p\,\Gamma(1+s)\quad m!}$$

$$\times\ {}_2F_3\left[\begin{matrix} -m, -s\,; \\ 1+\alpha, -p-s, -\beta-p-s\,; \end{matrix}\ -1\right],$$

where $L_m^{\alpha}(x)$ are Laguerre polynomials.

Analogous results involving odd polynomials $H_{2pk+1}(x;k)$ could be easily obtained from (2.3).

Again, if we put $k = 1$ then we shall get the results involving Hermite polynomials.

4. Expansion Formulae

We shall use the results of §2 to obtain some expansion problems. Take

$$
{}_{p'+\delta}F_q\left[\begin{matrix}\Delta(\delta,-m),\,(a_{p'});\\ (b_q);\end{matrix}\ \lambda x^{-2ck}\right] x^{2pk+2sk+(\delta-1)m} = \sum_{r=0}^{\infty} g_{2kr}\, H_{2kr}(x\,;k).
$$

In order to determine g_{2kr} multiply both the sides of this by

$$x^{2k-2}\exp(-x^{2k})\, H_{2pk}(x\,;k).$$

Integrating over $(-\infty,\infty)$ and using the orthogonality relation (Thakare and Karande 1973).

$$
\int_{-\infty}^{\infty} x^{2k-2}\exp(-x^{k})\,[H_{2pk}(x\,;k)]^2\,dx = \frac{-2\beta\,[(2p)!]^2\,\Gamma(1+\beta)}{p!\,(1+\beta)_p}
$$

we shall obtain, because of (2.2), the relation

$$
g_{2pk} = \frac{p!}{(2p)!\,(1+\beta)} \times \frac{\Gamma\left(1+p+s+\frac{(\delta-1)m}{2k}\right)\Gamma\left(1+p+s+\beta+\frac{(\delta-1)m}{2k}\right)}{\Gamma\left(1+s+\frac{(\delta+1)m}{2k}\right)}\ {}_{p'+\delta+c}F_{q+2c}
$$

$$
\times\left[\begin{matrix}\Delta(\delta,-m),\,\Delta\left(c,\frac{-2sk-(\delta-1)m}{2k}\right),\,(a_{p'})\,;\\ (b_q),\,\Delta\left(c,\frac{-2pk-2sk-(\delta-1)m}{2k}\right),\,\Delta\left(c,\frac{1-2pk-2sk-(\delta-1)m}{2k}\right);\end{matrix}\ \frac{\lambda(-1)^c}{c^c}\right].
$$

Therefore, we have

$$
x^{2pk+2sk}\left\{x^{(\delta-1)m}\,{}_{p'+\delta}F_q\left[\begin{matrix}\Delta(\delta,-m),\,(a_{p'})\,;\\ (b_q)\,;\end{matrix}\ \lambda x^{-2ck}\right]\right\}
= \frac{1}{\Gamma\left(1+s+\frac{(\delta-1)m}{2k}\right)\Gamma(1+\beta)}\times
$$

(*equation continued on p. 645*)

$$\times \sum_{r=0}^{\infty} \left\{ \frac{r!\, \Gamma\left(1 + r + s + \frac{(\delta - 1)m}{2k}\right)}{(2r)!} \right.$$

$$\times \Gamma\left(1 + r + s + \beta + \frac{(\delta - 1)m}{2k}\right) H_{2kr}(x; k)\ {}_{p'+\delta+c}F_{q+2c}$$

$$\times \left[\begin{matrix} \Delta(\delta, -m), \Delta\left(c, \frac{-2sk - (\delta - 1)m}{2k}\right), (a_{p'}); \\ \\ \Delta\left(c, \frac{-2rk - 2sk - (\delta - 1)m}{2k}\right), \Delta\left(c, \frac{1 - 2rk - 2sk - (\delta - 1)m}{2k}\right), (b_q); \end{matrix} \ \frac{\lambda(-1)^c}{c^c} \right] \quad \ldots(4.1)$$

The following special cases of (4.1) are worth noting :

(i) Put $\lambda = c = \delta = p' = q = 1$, $a_1 = 1 + \alpha + \beta + m$, and $b_1 = 1 + \alpha$. Multiplying both the sides by $(1 + \alpha)_m/m!$, we have

$$x^{2pk+2sk} P_m^{(\alpha, \beta)} (1 - 2x^{-2k}) = \frac{(1 + \alpha)_m}{m!\, \Gamma(1 + s)\, \Gamma(1 + \beta)}$$

$$\times \sum_{r=0}^{\infty} \frac{r!}{(2r)!} \Gamma(1 + r + s)\, \Gamma(1 + r + s + \beta)\, H_{2kr}(x\ ; k)$$

$$\times\ {}_3F_3 \left[\begin{matrix} -m, 1 + \alpha + \beta + m, -s\ ; \\ \\ 1 + \alpha\ , -p - s\ , \frac{1 - 2pk - 2sk}{2k}\ ; \end{matrix} \ -1 \right]$$

where $P_m^{(\alpha, \beta)}(x)$ are Jacobi polynomials. Similar result for odd polynomials $H_{2pk+1}(x\ ; k)$ holds. It is again to be noted that if we put $k = 1$, we shall get results involving Jacobi polynomials and Hermite polynomials. In particular, we have

$$x^{2p+2s} P_m^{(\alpha, \beta)}(1 - 2x^{-2}) = \frac{(1 + \alpha)_m}{m!\, \Gamma(1 + s)\, \Gamma(1 + \beta)} \sum_{r=0}^{\infty} \left\{ \frac{r!\, \Gamma(1 + r + s)}{(2r)!} \right.$$

$$\left. \times \Gamma(1 + r + s + \beta)\, H_{2r}(x) \right\}\ {}_3F_3 \left[\begin{matrix} -m, 1 + \alpha + \beta + m, -s\ ; \\ \\ 1 + \alpha, -p - s, \frac{1 - 2p - 2s}{2}\ ; \end{matrix} \ -1 \right].$$

(ii) Put $\delta = c = \lambda = q = 1, p' = 0$ and $b_1 = 1 + \alpha$ in (4.1). Multiplying both the sides by $\frac{(1+\alpha)_m}{m!}$ we obtain

$$x^{2pk+2sk} L_m^{\alpha}(x^{-2k}) = \frac{(1+\alpha)_m}{m!\,\Gamma(1+s)\,\Gamma(1+\beta)} \sum_{r=0}^{\infty} \frac{r!\,\Gamma(1+r+s)}{(2r)!} \cdot$$

$$\times \Gamma(1+r+s+\beta)\, H_{2kr}(x;k)\, {}_2F_3\left[\begin{matrix} -m, -s\,; \\ \\ 1+\alpha, -p-s, \dfrac{1-2pk-2sk}{2k}\,; \end{matrix} \; -1\right]$$

The case when $k = 1$, is also of interest.

$$x^{2p+2s} L_m^{\alpha}(x^{-2}) = \frac{(1+\alpha)_m}{m!\,\Gamma(1+s)\,\Gamma(1+\beta)} \sum_{r=0}^{\infty} \frac{r!\,\Gamma(1+r+s)}{(2r)!}$$

$$\times \Gamma(1+r+s+\beta)\, H_{2r}(x)\, {}_2F_3\left[\begin{matrix} -m, -s\,; \\ \\ 1+\alpha, -p-s, \dfrac{1-2p-2s}{2}\,; \end{matrix} \; -1\right].$$

(iii) Set $\delta = c = \lambda = 1$, $a_1 = 1 + \alpha + \beta + m$, $b_1 = 1 + \alpha$, $b_2 = \frac{1}{2}$ in (4.1) Multiplying both the sides by $\frac{(1+\alpha)_m}{m!}$, we have

$$x^{2pk+2sk} f_m^{(\alpha,\beta)}\left(\begin{matrix} a_2, \ldots, a_{p'}\,; \\ \\ b_3, \ldots, b_q\,; \end{matrix} \; x^{-2k}\right)$$

$$= \frac{1}{\Gamma(1+s)\,\Gamma(1+\beta)} \sum_{r=0}^{\infty} \left\{\frac{r!\,\Gamma(1+r+s)}{(2r)!}\, \Gamma(1+r+s+\beta)\right.$$

$$\times H_{2kr}(x;k)\, f_m^{(\alpha,\beta)}\left(\begin{matrix} a_2, a_3, \ldots, a_{p'}, -s\,; \\ \\ b_3, b_4, \ldots, b_q, -r-s, \dfrac{1-2rk-2sk}{2k}\,; \end{matrix} \; -1\right)$$

where $f_m^{(\alpha,\beta)}$ is the same as defined in (3.2).

Again if we put $p' = q = 3$, $a_2 = \rho$, $a_3 = \frac{1}{2}$, $b_3 = \sigma$ in the preceding relation, then we shall get the result involving the generalized Rice polynomials $f_m^{(\alpha,\beta)}(\rho, \sigma ; x)$ as defined by (3.3), i.e. we shall get

$$x^{2pk+2sk} f_m^{(\alpha,\beta)}(\rho, \sigma, x^{-2k}) = \frac{1}{\Gamma(1+s)\,\Gamma(1+\beta)}$$

$$\times \sum_{r=0}^{\infty} \frac{r!}{(2r)!} \Gamma(1+r+s)\,\Gamma(1+r+s+\beta)\, H_{2kr}(x ; k)$$

$$\times \frac{(1+\alpha)_m}{m!}\, {}_4F_4 \left[\begin{matrix} -m, 1+\alpha+\beta+m, \rho, -s ; \\ \sigma, 1+\alpha, -p-s, \dfrac{1-2pk-2sk}{2k} ; \end{matrix} \; -1 \right].$$

Similar result for the odd polynomials $H_{2pk+1}(x ; k)$ holds.

It may be noted that it is possible to give such results involving the polynomials of Bedient, Meixner and related polynomials.

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