SOME ASYMPTOTIC FORMULAE

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A second-order matrix differential equation $(L - \lambda F)\,\phi = 0$ in a finite interval with suitable boundary conditions at the end points has been considered and the asymptotic formulae for eigenvectors and Fourier coefficient s of a function, for large $|\lambda|$ have been deduced.

§1. Let L denote the matrix operator

$$L \equiv \begin{pmatrix} \dfrac{d}{dx}\left(p_0(x)\dfrac{d}{dx}\right) + p_1(x) & r(x) \\ r(x) & \dfrac{d}{dx}\left(q_0(x)\dfrac{d}{dx}\right) + q_1(x) \end{pmatrix} \qquad \ldots(1.1)$$

F the symmetric matrix

$$F = (F_{ij}) \equiv \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix}, \qquad \ldots(1.2)$$

and ϕ a vector having two components $u \equiv u(x)$ and $v \equiv v(x)$ represented by a column matrix

$$\phi = \begin{pmatrix} u \\ v \end{pmatrix}.$$

The equation to be considered is

$$(L - \lambda F)\,\phi = 0, \quad a \leqslant x \leqslant b, \qquad \ldots(1.3)$$

where λ is a parameter.

We assume that $p_0(x)$, $q_0(x)$, $p_1(x)$, $q_1(x)$, $r(x)$ and F satisfy all the conditions of §1 of Bhagat (1969) and the boundary conditions of §1 of Bhagat (1969) are also satisfied by any solution of (1.3). We use the notations of Bhagat (1969).

In this paper we are concerned with asymptotic formulae for eigenvectors and 'Fourier coefficients' of a function $f(x)$, for large $|\lambda|$. The method has been suggested by § 2.14 of Titchmarsh (1962).

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§2. *Theorem* 1—Let $\Psi_n(x, \lambda_n) = \begin{pmatrix} \Psi_{1n}(x) \\ \Psi_{2n}(x) \end{pmatrix}$ be eigenvectors. Then for fixed x,

$$\Psi_{1n}(x),\ \Psi_{2n}(x) = O(\,|\,\lambda_n\,|^{\,1/4}) \qquad \ldots(2.1)$$

if $\det F \geqslant 1$ in $a \leqslant x \leqslant b$.

We have, by integration by parts and eqn. (1.3)

$$\int_x^\xi (\xi - y)^2 (y - x) [\lambda_n(F_{11}\Psi_{1n}(y) + F_{12}\Psi_{2n}(y)) - p_1(y)\,\Psi_{1n}(y) - r(y)\,\Psi_{2n}(y)]\,dy$$

$$= (\xi - x)^2 p_0(x)\,\Psi_{1n}(x) + \int_x^\xi (6y - 2x - 4\xi)\,p_0(y)\,\Psi_{1n}(y)\,dy$$

$$+ \int_x^\xi \{(\xi - y)^2 - 2(\xi - y)(y - x)\}\,p_0'(y)\,\Psi_{1n}(y)\,dy. \qquad \ldots(2.2)$$

Let $A = \max\limits_{x \leqslant y \leqslant \xi} (\,|\,r(y)\,|\,,\,|\,p_0(y)\,|\,,\,|\,p_0'(y)\,|\,,\,|\,p_1(y)\,|\,,\,|\,F_{11}\,|\,,\,|\,F_{12}\,|\,,\,|\,F_{22}\,|\,)$.

Now

$$|\int_x^\xi (\xi - y)^2 (y - x) F_{11}\Psi_{1n}(y)\,dy\,| \leqslant (\xi - x)^3 A \int_x^\xi |\,\Psi_{1n}(y)\,|\,dy$$

$$\leqslant (\xi - x)^3 A\{\int_x^\xi dy \int_x^\xi |\,\Psi_{1n}(y)\,|^2\,dy\}^{1/2},$$

by Schwarz's inequality

$$\leqslant A(\xi - x)^{7/2}\,[\int_x^\xi \{\,|\,\Psi_{1n}(y)\,|^2 + |\,\Psi_{2n}(y)\,|^2\}$$

$\times\ dy]^{1/2}$, by Mirsky (1955, Prob. 37, p. 426)

$$\leqslant A(\xi - x)^{7/2} \left[\int_x^\xi \Psi_n^T(y)\,F\Psi_n(y)\,dy\right]^{\frac{1}{2}}$$

$$\leqslant A(\xi - x)^{7/2}.$$

Similarly

$$|\int_x^\xi (\xi - y)^2 (y - x) F_{12}\Psi_{2n}(y)\,dy\,| \leqslant A(\xi - x)^{7/2}$$

$$|\int_x^\xi (\xi - y)^2 (y - x)\{p_1(y)\,\Psi_{1n}(y) + r(y)\,\Psi_{2n}(y)\}\,dy\,| \leqslant 2A(\xi - x)^{7/2}$$

$$|\int_x^\xi (6y - 2x - 4\xi)\,p_0(y)\,\Psi_{1n}(y)\,dy\,| \leqslant 2A(\xi - x)^{3/2}$$

$$|\int \{(\xi - y)^2 - 2(\xi - y)(y - x)\}\,p_0'(y)\,\Psi_{1n}(y)\,dy\,| \leqslant A(\xi - x)^{5/2}.$$

Hence from (2.2), on putting $\xi = x + |\lambda_n|^{-1/2}$, we have

$$|\Psi_{1n}(x)| \leqslant \frac{A}{B}(2|\lambda_n|^{1/4} + 2|\lambda_n|^{-3/4} + |\lambda_n|^{-1/4} + 2|\lambda_n|^{1/4},$$

where $B = |p_0(x)|$ i.e. $\Psi_{1n}(x) = O(|\lambda_n|^{1/4})$.

Similarly

$$\Psi_{2n}(x) = O(|\lambda_n|^{1/4}).$$

Theorem 2—If the conditions of the Theorem 1 are satisfied

$$\Psi'_{1n}(x),\ \Psi'_{2n}(x) = O(|\lambda_n|^{3/4}). \qquad \ldots(2.3)$$

We have, by integration by parts and eqn. (1.3)

$$\int_x^\xi (\xi - y)^2 [\lambda_n\{F_{11}\Psi_{1n}(y) + F_{12}\Psi_{2n}(y)\} - p_1(y)\Psi_{1n}(y) - r(y)\Psi_{2n}(y)]\, dy$$

$$= -(\xi - x)^2 p_0(x)\Psi'_{1n}(x) - 2(\xi - x) p_0(x)\Psi_{1n}(x)$$

$$+ 2\int_x^\xi p_0(y)\Psi_{1n}(y)\, dy - 2\int_x^\xi (\xi - y) p_0'(y)\Psi_{1n}(y)\, dy. \qquad \ldots(2.4)$$

Just as in Theorem 1

$$\left|\int_x^\xi (F_{11}\Psi_{1n} + F_{12}\Psi_{2n})(\xi - y)^2\, dy\right| \leqslant 2A(\xi - x)^{5/2}$$

$$\left|\int_x^\xi (p_1\Psi_{1n} + r\Psi_{2n})(\xi - y)^2\, dy\right| \leqslant 2A(\xi - x)^{5/2}$$

$$\left|\int_x^\xi p_0(y)\Psi_{1n}(y)\, dy\right| \leqslant A(\xi - x)^{1/2}$$

$$\left|\int_x^\xi (\xi - y)p_0'(y)\Psi_{1n}(y)\, dy\right| \leqslant A(\xi - x)^{3/2}.$$

Therefore, from (2.4)

$$|\Psi'_{1n}(x)| \leqslant \frac{A}{B}\{6|\lambda_n|(\xi - x)^{1/2} + 6(\xi - x)^{1/2} + 4(\xi - x)^{-1/2} + 6(\xi - x)^{-3/2}\}.$$

Putting $\xi = x + |\lambda_n|^{-1/2}$, it follows that

$$\Psi'_{1n}(x) = O(|\lambda_n|^{3/4}).$$

Similarly

$$\Psi'_{2n}(x) = O(|\lambda_n|^{3/4}).$$

Theorem 3—Let $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ be a vector which has continuous derivatives up to order four and which together with its derivatives up to order three vanish at $x = a$ and $x = b$, $p_0(x)$, $q_0(x)$ have continuous derivatives up to third order, $p_1(x)$, $q_1(x)$, $r(x)$, F_{ij} have continuous second order derivatives in $a \leqslant x \leqslant b$ and let det $F \geqslant 1$. Let

$$c_n = \int_a^b \Psi_n^T Ff\, dx$$

be the 'Fourier coefficient' of $f(x)$, then

$$c_n = O\left(\frac{1}{\lambda_n^2}\right),$$ λ_n being the eigenvalue corresponding to the eigenvector $\Psi_n(x)$.

From (1.3) we have

$$\Psi_{1n}(x) = \frac{(r-\lambda_n F_{12})(q_0\Psi'_{2n})' - (q_1 - \lambda_n F_{22})(p_0\Psi'_{1n})'}{H}, \qquad \text{...(2.5)}$$

$$\Psi_{2n}(x) = \frac{(r-\lambda_n F_{21})(p_0\Psi'_{1n})' - (p_1 - \lambda_n F_{11})(q_0\Psi'_{2n})'}{H}, \qquad \text{...(2.6)}$$

where $H = (F_{11}F_{22} - F_{12}^2)\lambda_n^2 - (p_1F_{22} + q_1F_{11} - 2rF_{12})\lambda_n + p_1q_1 - r^2$, which does not vanish for large values of $|\lambda_n|$, for, $F_{11}F_{22} - F_{12}^2 > 0$.

By (1.3)

$$\int_a^b (F_{11}f_1\Psi_{1n} + F_{12}f_1\Psi_{2n})\, dx = \int_a^b f_1 \cdot \frac{(p_0\Psi'_{1n})' + p_1\Psi_{1n} + r\Psi_{2n}}{\lambda_n} dx$$

$$= \int_a^b \frac{f_1(p_0\Psi'_{1n})'}{\lambda_n} dx + \int_a^b \frac{f_1}{\lambda_n} \cdot \frac{\{r(r-\lambda_n F_{21}) - p_1(q_1 - \lambda_n F_{22})\}(p_0\Psi'_{1n})'}{H} dx$$

$$+ \int_a^b \frac{f_1}{\lambda_n} \cdot \frac{\{p_1(r-\lambda_n F_{12}) - r(p_1 - \lambda_n F_{11})(q_0\Psi'_{2n})'}{H} dx,$$ by (2.5) and (2.6)

$$= I_1 + I_2 + I_3, \text{ say.} \qquad \text{...(2.7)}$$

Now

$$I_2 = \int_a^b \frac{f_1}{\lambda_n} \cdot \frac{(r^2 - p_1 q_1)(p_0 \Psi'_{1n})'}{H} dx + \int_a^b \frac{f_1(p_1 F_{22} - rF_{21})(p_0 \Psi'_{1n})'}{H} dx$$

$$= I_{21} + I_{22}, \text{ say.}$$

$$I_{22} = \left[\frac{f_1(p_1 F_{22} - rF_{21})}{H} p_0 \Psi'_{1n} - p_0 \frac{d}{dx} \left\{ \frac{f_1(p_1 F_{22} - rF_{21})}{H} \right\} \Psi_{1n} \right]_a^b$$

$$+ \int_a^b \Psi_{1n} \cdot \frac{d}{dx} \left[p_0 \frac{d}{dx} \left\{ \frac{f_1(p_1 F_{22} - rF_{21})}{H} \right\} \right] dx.$$

The first term in I_{22} vanishes from initial conditions on f.

By Schwarz's inequality and using $\Psi_n^T F \Psi_n \geqslant \Psi_n^T \Psi_n$,

$$\int_a^b \Psi_{1n}(x)\, dx = O(1).$$

$$\frac{d}{dx}\left[p_0 \frac{d}{dx} \left\{ \frac{f_1(p_1 F_{22} - rF_{21})}{H} \right\} \right] = O\left(\frac{1}{\lambda_n^2} \right).$$

Hence

$$I_{22} = O\left(\frac{1}{\lambda_n^2} \right). \qquad \ldots(2.8)$$

Similarly

$$I_{21} = O\left(\frac{1}{\lambda_n^3} \right). \qquad \ldots(2.9)$$

Thus

$$I_2 = O\left(\frac{1}{\lambda_n^2} \right). \qquad \ldots(2.10)$$

Similarly

$$I_3 = O\left(\frac{1}{\lambda_n^2} \right). \qquad \ldots(2.11)$$

Now

$$I_1 = \frac{1}{\lambda_n} \left[f_1 p_0 \Psi'_{1n} - f'_1 p_0 \Psi_{1n} \right]_a^b + \frac{1}{\lambda_n} \int_a^b \frac{r(p_0 f'_1)'}{H} (q_0 \Psi'_{2n})'\, dx -$$

(*equation continued on p.* 653)

$$- \int_a^b \frac{(p_0 f_1')' F_{12}}{H} (q_0 \Psi_{2n}')' \, dx - \frac{1}{\lambda_n} \int_a^b \frac{q_1 (p_0 f_1')'}{H} (p_0 \Psi_{1n}')' \, dx$$

$$+ \int_a^b \frac{(p_0 f_1')' F_{22}}{H} (p_0 \Psi_{1n}')' \, dx, \text{ by (2.5)}$$

$$= I_{11} + I_{12} + I_{13} + I_{14} + I_{15}, \text{ say} \qquad \ldots(2.12)$$

$$I_{15} = \left[\frac{p_0 F_{22} (p_0 f_1')'}{H} - p_0 \frac{d}{dx} \left\{ \frac{F_{22}(p_0 f_1')'}{H} \right\} \Psi_{1n} \right]_a^b$$

$$+ \int_a^b \Psi_{1n} \frac{d}{dx} \left[p_0 \frac{d}{dx} \left\{ \frac{F_{22}(p_0 f_1')'}{H} \right\} \right] dx.$$

The first term in I_{15} is zero by initial conditions on f and $\int_a^b \Psi_{1n}(x)\, dx = O(1)$.

$$\frac{d}{dx} \left[p_0 \frac{d}{dx} \left\{ \frac{F_{22}(p_0 f_1)'}{H} \right\} \right] = O\left(\frac{1}{\lambda_n^2} \right).$$

Thus

$$I_{15} = O\left(\frac{1}{\lambda_n^2} \right).$$

Similarly

$$I_{13} = O\left(\frac{1}{\lambda_n^2} \right)$$

$$I_{12} = O\left(\frac{1}{\lambda_n^3} \right)$$

$$I_{14} = O\left(\frac{1}{\lambda_n^3} \right)$$

and by initial conditions on f

$$I_{11} = 0.$$

Hence from (2.12) we get

$$I_1 = O\left(\frac{1}{\lambda^2} \right). \qquad \ldots(2.13)$$

Thus by (2.10), (2.11) and (2.13)

$$\int_a^b f_1(F_{11}\Psi_{1n} + F_{12}\Psi_{2n})\,dx = O\left(\frac{1}{\lambda_n^2}\right). \qquad \text{...(2.14)}$$

In the same way

$$\int_a^b f_2(F_{21}\Psi_{1n} + F_{22}\Psi_{2n})\,dx = O\left(\frac{1}{\lambda_n^2}\right). \qquad \text{...(2.15)}$$

Adding (2.14) and (2.15), the result follows.

References

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