ON SELECTION OF PREDICTORS FOR A GIVEN CORRECTOR IN THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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In this paper we study the nature of the extraneous roots of a predictor-corrector set in PECE form and thereby indicate how a proper predictor can be chosen for a given corrector, depending upon the nature of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. We also study as to how the unstability in the corrector (when it exists) can be removed by proper selection of predictor.

1. INTRODUCTION

Consider a k-step scheme

$$\rho(E)\, y_n - h\sigma(E)\, f_n = 0 \qquad \ldots(1.1)$$

where $\rho(z) : = \sum_{\nu=0}^{k} \alpha_\nu y_{n+\nu}$,

$$\sigma(z) : = \sum_{\nu=0}^{k} \beta_\nu f_{n+\nu},\ Ey_n : y_{n+1}$$

and $f_n : = f(x_n, y_n)$. (1.1) is called D-stable or stable for $h \to 0$ if all the zeros of ρ lie in $| z | \leqslant 1$ and zeros with $| z | = 1$ are simple. (1.1) is said to be of order p if for sufficiently differentiable function $y(x)$,

$$\rho(E_h)\, y(x) - h\sigma(E_h)\, y'(x) = O\,(h^{p+1}) \qquad \ldots(1.2)$$

where $E_h y(x) : = y(x + h)$. Let (1.1) satisfy consistency conditions viz $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$ (Henrici 1962, p. 224). Consider the initial value problem

$$y' = \lambda y,\ y(0) = 1 \qquad \ldots(1.3)$$

the exact solution of which is $y(x) = e^{\lambda x}$. For the problem (1.3), the scheme (1.1) reduces to

$$\rho(E)\, y_n - H\sigma(E)\, y_n = 0 \qquad \ldots(1.4)$$

where $H = \lambda h$. The characteristic equation corresponding to the difference equation (1.4) is,

$$\phi(z, H) = \rho(z) - H\sigma(z) \qquad \ldots(1.5)$$

We denote the zeros of $\phi(z, H)$ by $\xi_\nu(H)$, $\nu = 1(1)\,k$. We know from Henrici (1962, p. 238) that there exists a zero of $\phi(z, H)$, termed as 'principal zero' to be denoted by $\xi_1(H)$ which satisfies

$$\xi_1(H) = e^H + O(H^{p+1}). \qquad \ldots(1.6)$$

For a given value of H (real), we shall call a D-stable scheme (1.1) strongly stable (Stetter 1965, p. 84) if

$$|\,\xi_\nu(H)\,| \leqslant \xi_1(H),\ \nu = 2(1)\,k. \qquad \ldots(1.7)$$

Each D-stable scheme (1.1) is strongly stable for $H = 0$. By continuity, there will be a largest number $H^+ \geqslant 0$ and a smallest number $H^- \leqslant 0$, such that the scheme is strongly stable for all $H \in [H^-, H^+]$. The interval $[H^-, H^+]$ is termed as 'stability interval'.

For a D-stable scheme (1.1), the zeros of $\rho(z)$ with modulus 1 are termed as essential zeros and the other zeros are termed as non-essential zeros. For zeros $\xi_\nu(H)$ which approach essential zeros, we have from Henrici (1962, p 237),

$$\xi_\nu(H) = \xi_{\nu 0}(1 + a\,H + O(H^2)) \qquad \ldots(1.8)$$

where

$$a_\nu = \frac{\sigma(\xi_{\nu 0})}{\xi_{\nu 0}\rho'(\xi_{\nu 0})},\quad \nu = 1(1)\,s$$

$\xi_{\nu 0} = \xi_\nu(0)$ are the zeros of $\rho(z)$ and s is the number of essential zeros. From the consistency conditions for (1.1), it follows that

$$a_1 = \frac{\sigma(1)}{(1)\,\rho'(1)} = 1.$$

2. Analysis

(A) Now we shall show that even when $\xi_\nu(H)$ does not approach an essential zero, a relation of the type (1.8) holds. By Maclaurin series we find that $\xi_\nu(H)$ must be of the form

$$\xi_\nu(H) = \xi_{\nu 0} + c_\nu H + O(H^2),\ \ \nu = s + 1(1)\,k. \qquad \ldots(2.1)$$

Substituting (2.1) in (1.5), expanding the polynomials ρ and σ about the point $\xi_{\nu 0}$ and retaining the terms upto first order of H we have,

$$\rho(\xi_{\nu 0}) + c_\nu H\rho'(\xi_{\nu 0}) - H(\xi_{\nu 0}) = 0.$$

As $\rho(\xi_{\nu 0}) = 0$, from above equation we get,

$$c_\nu = \frac{\sigma(\xi_{\nu 0})}{\rho'(\xi_{\nu 0})},\quad \nu = s + 1(1)\,k. \qquad \ldots(2.2)$$

For $\xi_{\nu 0} \neq 0$, we define

$$a_\nu = \frac{c_\nu}{\xi_{\nu 0}} = \frac{\sigma(\xi_{\nu 0})}{\xi_{\nu 0}\rho'(\xi_{\nu 0})}, \; \nu = s + 1(1)\, k \qquad \text{...(2.2}a\text{)}$$

as the growth parameter corresponding to $\xi_{\nu 0}$. Substituting c_ν in (2.1) and using (2.2*a*), we find that (2.1) is of the form (1.8). We give two examples below to illustrate this.

Example 1 — Consider the scheme

$$\rho(z) = z^2 - (4/5)\, z - 1/5$$

$$\sigma(z) = (2/5)\, z^2 + 4z/5.$$

The zeros of $\rho(z)$ are $\xi_{10} = 1$ and $\xi_{20} = -\frac{1}{5}$. ξ_{20} is not an essential zero.

$$a_2 = \frac{\sigma\left(-\frac{1}{5}\right)}{\left(-\frac{1}{5}\right)\rho'\left(-\frac{1}{5}\right)} = -\frac{3}{5}.$$

A simple algebraic calculation shows that

$$\xi_2(H) = \left(-\tfrac{1}{5}\right)\left(1 - \tfrac{3}{5}\, H + O\,(H^2)\right)$$
$$= \left(-\tfrac{1}{5}\right)\left(1 + a_2 H + O\,(H^2)\right)$$

In particular when $\xi_{\nu 0} = 0$, $\nu > 1$, from (2.1), $\xi_\nu(H)$ must be of the form

$$\xi_\nu(H) = c_\nu H + O\,(H^2).$$

Example 2 — Consider the scheme $\rho(z) = z^2 - z$

$$\sigma(z) = \frac{5}{12}\, z^2 + \frac{2}{3}\, z - \frac{1}{12}.$$

Here $\xi_{20} = 0$, $c_2 = \dfrac{\sigma(0)}{\rho'(0)} = \dfrac{-\frac{1}{12}}{(-1)} = \dfrac{1}{12}$ and we then have

$$\xi(H) = \frac{1}{12}\, H + O\,(H^2).$$

This can be easily seen to be so.

(B) Considering only first two terms of $\xi_\nu(H)$, we have

$$\xi_\nu(H) \approx \xi_{\nu 0}\, e^{a_\nu H}, \; \nu = 1(1)\, k. \qquad \text{...(2.3)}$$

We know that (Henrici 1962, p. 236) the solution of the difference equation (1.4) is

$$y_n = \sum_{\nu=1}^{k} c_\nu (\xi_\nu(H))^n \qquad \text{...(2.4)}$$

where the constants c_ν can be determined from the initial conditions. From Henrici (1962, p. 239), we have $c_1 = 1 + O(H^2)$. Using (1.6) and (2.3), (2.4) can be written as

$$y_n = e^{Hn} + \sum_{\nu=2}^{k} c_\nu(\xi_{\nu 0})^n e^{a_\nu H n} + \text{higher order terms in } H. \quad \ldots(2.5)$$

The first term in (2.5) viz, e^{Hn} grows like the solution of (1.3) at $x = x_n$. In order that the scheme (1.1) be used in the integration of (1.3), it is necessary that the sum, viz, $\sum_{\nu=2}^{k} c_\nu(\xi_{\nu 0})^n e^{a_\nu H n}$ in (2.5) decays or atleast grows less strongly than the first term.

Now consider the predictor corrector scheme

$$y_{n+k}^{(o)} = -\sum_{\nu=0}^{k-1} \alpha_\nu^* y_{n+\nu} + h \sum_{\nu=0}^{k-1} \beta_\nu^* f_{n+\nu}$$

$$y_{n+k}^{(i)} = -\sum_{\nu=0}^{k} \alpha_\nu y_{n+\nu} h \left[\sum_{\nu=0}^{k-1} \beta_\nu f_{n+\nu} + \beta_k f\left(x_{n+k}, y_{n+k}^{(i-1)}\right)\right] \quad \ldots(2.6)$$

$\beta_k \neq 0$, $i = 1(1)\, m$.

For $\rho(EC)^m E$ algorithm the scheme (2.6) transforms the polynomial (1.5) into (Stetter 1965, p. 85)

$$\phi^m(z, H) = [(1 - B^m)(\rho(z) - \sigma(z)) + B^m(1 - B)(\rho^*(z) - \sigma^*(z))] \quad \ldots(2.7)$$

where $B = H\beta_k$; $\rho^*(z) = \sum_{\nu=0}^{k} \alpha_\nu^* z^\nu$, $\alpha_k = 1$ and $\overset{*}{\sigma}(z) = \sum_{\nu=0}^{k-1} \beta_\nu z^\nu$. Assuming that the predictor is of order $q > 0$, it is clear from (1.5), (1.6) and (2.7) that the zeros $\xi_\nu^m(H)$ of $\phi^m(z, H)$ satisfy

$$\left.\begin{aligned} \xi_1^m(H) &= e^H + O(H^{p+1}) + O(H^{q+m+1}) \\ \xi_\nu^m(H) &= \xi_\nu(H) + O(H^m), \quad \nu = 2(1)\, k. \end{aligned}\right\} \quad \ldots(2.8)$$

Any sort of unstability which may occur in the corrector is due to violation of (1.7) which is just a first order effect in H. As such, we consider the case when $m = 1$ (i.e. the corrector is used only once). For simplicity we shall denote the zeros of $\phi^1(z, H)$ by $\xi_\nu(H)$, $\nu = 1(1)\, k$. Let

$$\xi_\nu(H) = (\xi_{\nu 0})\,[1 + A_\nu H + B_\nu H^2 + O(H^3)] \quad \ldots(2.9)$$

We find (Stetter 1965, p. 86)

$$\xi_\nu(H) = \xi_{\nu 0} + \frac{\tau_\nu}{\rho'_\nu} + H^2\left[\left(-\rho''_\nu\,\tau_\nu^2 / 2\rho'_\nu\right) + \tau_\nu\tau'_\nu + \beta_k\rho'_\nu\,\sigma^*_\nu\right]$$

$$\times \frac{1}{\rho'^2_\nu} + O(H^3) \qquad \text{...(2.10)}$$

where $\tau(z) = \sigma(z) - \beta_k\,\rho^*(z)$, a dash denotes differentation w.r.t. z, and $\rho_\nu = \rho(\xi_{\nu 0})$, $\tau_\nu = \tau(\xi_{\nu 0})$ etc.

Now we choose a real number d such that $\xi_\nu(H)$ satisfies :

$$|\,\xi_\nu(H)\,| \leqslant |\,\xi_{\nu 0}\,|\,e^{dH} \leqslant e^{dH} \leqslant e^{H} = \xi_\nu(H) \qquad \text{...(2.11)}$$

As $|\,\xi_{\nu 0}\,| \leqslant 1$, $|\,\xi_{\nu 0}\,|\,e^{dH} \leqslant e^{dH}$.

Now

$$e^{dH} \leqslant e^{H} \Leftrightarrow \left.\begin{array}{ll}\text{(i)} & d \leqslant 1 \text{ for } H > 0\\ \text{(ii)} & d \geqslant 1 \text{ for } H < 0\end{array}\right\} \qquad \text{...(2.12)}$$

(2.11) automatically holds for $H = 0$. Condition (2.11) or (2.12) is sufficient for strong stability defined by (1.7). When the condition (2.11) is applied to (2.9), we find

$$A_\nu = d \qquad \text{...(2.13}a\text{)}$$

and

$$B_\nu \leqslant \tfrac{1}{2}\,d^2 \qquad \text{...(2.13}b\text{)}$$

Conditions (2.13) when applied to (2.10) yield (taking $\xi_{\nu 0} = e^{i\omega\nu}$)

$$\text{Re}\left\{e^{-i\omega\nu}\,\frac{\tau_\nu}{\rho'_\nu}\right\} = d \qquad \text{...(2.14}a\text{)}$$

and

$$\text{Re}\left[e^{-i\omega\nu}\left\{\frac{\rho''_\nu}{2\rho'_\nu}\,\tau_\nu^2 + \tau_\nu\,\tau'_\nu + \beta_k\,\rho'_\nu\,\sigma^*_\nu\right\} / \rho'^2_\nu\right] \leqslant \tfrac{1}{2}d^2. \qquad \text{...(2.14}b\text{)}$$

Remark 1 : The conditions (2.12) will be strictly valid only when $\xi_\nu(H)$ grows exactly like $|\,\xi_{\nu 0}\,|\,e^{dH}$. It may happen that $B_\nu < \frac{1}{2}d^2$ so that $|\,\xi_\nu(H)\,| < |\,\xi_{\nu 0}\,|\,e^{dH}$. In that case the bounds on d in (2.12) are slightly relaxed. When $H > 0$, d can be slightly above 1 but not much above 1 and similarly when $H < 0$, d can be slightly below 1 (and never negative) but not much below 1. We shall see in §3 how these conditions are also enough to restore stability in the corrector.

Remark 2 : (2.14*a*) is a linear relation between coefficients of ρ^* and d. Thus we can determine a predictor (ρ^*, σ^*) which when combined with the corrector (ρ, σ) would give $| \xi_\nu(H) | \leqslant | \xi_{\nu 0} | e^{dH}$. The condition (2.14*b*) merely serves to put restriction on free parameters in ρ^*. When $d \neq 1$, the r.h.s. of (2.14*b*) can be replaced by $\frac{1}{2}$, but a strict inequality is needed when $d = 1$.

Remark 3 — The correctors for which $| \xi_{\nu 0} | < 1$, $\nu = 2(1)\, k$, can be easily seen to be strongly stable and in those cases conditions (2.12) are not of much help. However we find a rough bound for d for a particular choice of H. In order that the extraneous root which we have made to satisfy (2.11) decays, we must have that its contribution in absolute value to the sum in (2.5) at nth step should be greater than or equal to that at $(n + 1)$th step, i.e. we must have,

$$| C_\nu | \; | \xi_{\nu 0} |^{n} . \, e^{dHn} \geqslant | C_\nu | \; | \xi_{\nu 0} |^{n+1} . \, e^{dH(n+1)}$$

or
$$1 \geqslant | \xi_{\nu 0} | . \, e^{dH}$$

or
$$\log_e \left(\frac{1}{| \xi_{\nu 0} |} \right) \geqslant dH \text{ for } \xi_{\nu 0} \neq 0 \qquad \ldots(2.15)$$

(2.15) can be used in place of (2.12) for strongly stable correctors.

This discussion applies directly to the initial value problem in the general form, viz. $y' = f(x, y)$, $y(x_0) = y_0$, with $f_y(x, y(x))$ replacing λ.

3. APPLICATIONS

In this section we study the effect of the choice of d both in the case of unstable and strongly stable correctors.

(A) We first consider an unstable corrector. We also indicate below how the unstability in the corrector can be eliminated. Consider Milne's corrector for which

$$\left. \begin{aligned} \rho(z) &= z^2 - 1 \\ \sigma(z) &= \frac{z^2}{3} + \frac{4}{3} z + \frac{1}{3} . \end{aligned} \right\} \qquad \ldots(3.1)$$

It is well known (Henrici 1962, p. 241) that (3.1) is an unstable corrector with $\xi_{20} = -1$ and $a_2 = -\frac{1}{3}$. All the predictor given by

$$\begin{aligned} y_{n+4} &= A_0 y_{n+3} + A_1 y_{n+2} + A_2 y_{n+1} + A_3 y_n \\ &\quad + h\, [B_0 f_{n+3} + B_1 f_{n+2} + B_2 f_{n+1}] \\ &\quad + E_5 \frac{h^5}{5} y^{(5)} (\xi) \end{aligned} \qquad \ldots(3.2)$$

where
$$A_0 = -8 - A_2 + 8A_3$$
$$A_1 = 9 - 9A_3$$

$$B_0 = \frac{17 + A_2 - 9A_3}{3}$$

$$B_1 = \frac{14 + 4A_2 - 18A_3}{3}$$

$$B_2 = \frac{-1 + A_1 + 9A_3}{3}$$

$$E_5 = \frac{40 - 4A_2 + 72A_3}{3}$$

are 4 step predictors of order 4 (Hamming 1962, p. 202). Substituting (3.1) and (3.2) in (2.14*a*), we get

$$[\{(-2/3) - (16/3)(A_3 - 1)\}/(-1)(-2)] = d$$

$$\text{or} \quad d = [(7 - 8A_3)/3]. \qquad \ldots(3.3)$$

We must choose d in accordance with (2.12) within the bounds dictated by the considerations of truncation and round-off errors. To keep truncation error low, E_5 is kept small and to minimize round off noise amplification (Hamming 1962, p. 202) the quantity

$$N_a = (A_0^2 + A_1^2 + A_2^2 + A_3^2)^{1/2}$$

is kept as small as possible. The free parameter A_2 is used for these purposes.

Case (i) : $H > 0$ — It follows from (2.12) that $d \leqslant 1$. For the fast decay of the extraneous root we must have d as small as possible. We find from (3.3) and (3.2) that E_5 increases as d decreases. Let us consider for illustration 4 predictors which when combined with (3.1) would give $d = 13/9, 1, -1/3$ and -1. We fix $A_2 = 0$ in all these formulae to keep N_a small.

(α) Let $A_2 = 0$, $d = 13/9$, from (3.3) $A_3 = 1/3$, $N_a = \sqrt{41}$.

Now the predictor (3.2) becomes

$$y_{n+4} = -\frac{16}{3} y_{n+3} + 6y_{n+2} + \frac{1}{3} y_n + \frac{h}{3}(14f_{n+3} + 8f_{n+2} + 2f_{n+1}) + \frac{64}{360} h^5 y^{(5)}(\xi). \qquad \ldots(3.4)$$

(β) Let $A_2 = 0$, $d = 1$, from (3.3) $A_3 = \frac{1}{2}$, $N_a = \sqrt{19}$.

Now (3.2) becomes

$$y_{n+4} = -4y_{n+3} + 4.5y_{n+2} + 0.5y_n + \frac{h}{6}(25f_{n+3} + 10f_{n+2} + 7f_{n+1}) + \frac{76}{360} h^5 y^{(5)}(\xi). \qquad \ldots(3.5)$$

(γ) $A_2 = 0$, $d = -1/3$, then 3.3 gives $A_3 = 1$, $N_a = 1$.

Now 3.2 becomes

$$y_{n+4} = y_n + \frac{4h}{3}[2f_{n+3} - f_{n+2} + 2f_{n+1}] + \frac{112}{360}h^5y^{(5)}(\xi). \qquad \ldots(3.6)$$

(δ) Let $A_2 = 0$, $A = -1$, then (3.3) gives $A_3 = 1$, $N_a = \sqrt{10.65}$

Now (3.2) becomes

$$y_{n+4} = 2y_{n+3} - \frac{9}{4}y_{n+2} + \frac{5}{4}y_n + \frac{h}{12}(23f_{n+3} - 34f_{n+2} + 44f_{n+1}) + \frac{180}{360}h^5y^{(5)}(\xi). \qquad \ldots(3.6a)$$

It is easy to verify (2.14*b*) in all these cases. We observe that d goes on decreasing as we pass from predictors given in (α) to that in (δ) [when we combine these predictors with (3.1)]. If we solve (1.3) with $H > 0$ with these $(p - c)$ sets, we expect the results to improve as we pass from predictor (α) to (δ).

This would have invariably happened for all the choices of h, if the local truncation errors of all these four predictors were same. But we observe that the local truncation error goes on increasing as we pass on from (α) to (δ). Thus only for sufficiently small h's we would get expected results. As h increases the truncation error will be dominant and we may not get the results as per our expectations. The Tables I–IV show the results for the solution of $y' = y$; $y(0) = 1$, using all these four predictors with (3.1) and taking $h = 0.01$, 0.05, 0.2 and 0.5 respectively. In Table 1, where $h = 0.01$ (quite small) we find the results of (3.1, (δ)) are better when compared to other four combinations [inspite of the fact that (δ) has the largest truncation among these four predictors (α) to (δ)]. As h increases, the truncation error will play an important role and the predictors with low truncation errors will be more useful. We observe that (γ) is better when $h = 0.05$ (Table II), (β) is better when $h = 0.2$ (Table III) and (α) is better when $h = 0.5$ (Table IV). The usage of (α) where $d > 1$ is justified because of the Remark 1 at the end of §2.

Case (ii) : $H < 0$ — From (2.12) it follows that $d \geqslant 1$. The choice of predictor with $d \geqslant 1$ for (3.1) which is unstable, would automatically remove unstability. The reason why the Milne-Simpson Scheme (3.1) and (3.6) cannot be used when $H < 0$ is that in this combination $d = -\frac{1}{3} < 1$.

Greater the d is chosen the better the results will be (of course keeping in mind N_a and E_5). As d in (3.1, 3.4) combination is greater than the d in (3.1, 3.5) combination, the results of set (3.1, 3.4) should be better than those of the other combinations when we solve problem (1.3) with $H < 0$. This is seen to be so from the Table V.

TABLE I

$y' = y$; $y(0) = 1$ *with* $h = 0.01$; *exact solution is* $y = e^x$

x	Relative error $\times 10^{14}$			
	(3.1, α)	(3.1, β)	(3.1, γ)	(3.1, δ)
0.2	− 887061	926	27086	− 882
0.4	− 1872744	1956	57211	− 1864
0.6	− 2858632	2985	87370	− 2848
0.8	− 3844741	4014	117553	− 3833
1.0	− 4831093	5043	147757	− 4819
1.2	− 5817708	6704	177975	− 5806
1.4	− 6804609	7102	208205	− 6793
1.6	− 7791822	8131	238443	− 7780
1.8	− 8779373	9161	268685	− 8768
2.0	− 9767293	10190	298937	− 9758

TABLE II

$y' = y$; $y(0) = 1$ *with* $h = 0.05$

x	(− Relative error $\times 10^{13}$)			
	(α)	(β)	(γ)	(δ)
2.0	72260	41246	11163	16038
4.0	216119	122965	33282	47916
6.0	360069	204782	55553	80198
8.0	504112	286682	77936	112753
10.0	648256	368658	100405	145472
12.0	792508	450702	122939	178311
14.0	936874	532808	145522	221224
16.0	1081364	614969	168143	244193
18.0	1225984	697182	190792	285209
20.0	1370746	779439	213463	310230

(α), (β), (γ), (δ) belong to Section 3 (A).

We notice that the results of the combinations (3.1, 3.4) and (3.1, 3.5) are better than those of the Milne Simpson method given by (3.1, 3.6) when $H > 0$ and h is

TABLE III

$y' = y; \; y(0) = 1$ *with* $h = 0.2$

x	Relative error $\times 10^{10}$			
	(α)	(β)	(γ)	(δ)
2.0	−11066	7580	64670	96222
4.0	−24868	17086	149729	223882
6.0	−38697	26651	235186	351753
8.0	−52556	36248	320663	479625
10.0	−66451	45865	406139	607496
12.0	−80389	55494	491615	735431
14.0	−94378	65128	577091	863232
16.0	−108427	74766	662565	991098

TABLE IV

$y' = y; \; y(0) = 1$ *with* $h = 0.5$

x	Relative error $\times 10^{8}$			
	(α)	(β)	(γ)	(δ)
2.0	17367	23129	40412	49048
4.0	51938	70673	141161	186833
6.0	87668	120102	244820	325031
8.0	122536	170153	348454	463042
10.0	158205	220401	451983	600862
12.0	193994	270700	555406	738492
14.0	229852	320999	658720	875430
16.0	265747	371283	761927	1013178
18.0	301660	421543	865028	1150236
20.0	337579	471780	968021	1287105

large and the results are comparable when h is small. Moreover (3.1, 3.6) cannot be used when $H < 0$. Also the number of function evaluations is same, namely 3, in all these three methods. Thus the use of either (3.4) or (3.5) as a predictor to (3.1) is preferred to (3.6).

TABLE V

$y' = -y; \quad y(0) = 1$; *exact solution is* $y = e^{-x}$

x	Relative error $\times 10^{10}$		Relative error $\times 10^{8}$	
	(3.1, 3.4)	(3.1, 3.5)	(3.1, 3.4)	(3.1, 3.5)
	$h = 0.2$		$h = 0.5$	
2.0	422669	459827	295290	349961
4.0	986840	1052260	1876380	1168800
6.0	1555111	1653401	4109789	3607676
8.0	2123678	2256607	6601270	5612064
10.0	2692239	2860433	9163539	7697715
12.0	3260769	3464426	11720299	9741286
14.0	3829268	4068447	14234458	11765000
16.0	4397734	4672451	16690613	13741167

(B) We now consider a strongly stable corrector.

$$\rho(z) = z^2 - \tfrac{1}{2}z - \tfrac{1}{2},$$
$$\sigma(z) = \tfrac{3}{8}z^2 + z + \tfrac{1}{8}, \qquad T_\gamma = \tfrac{3}{8}h^4 y^{(4)}(\xi). \qquad \ldots(3.7)$$

Here $\xi_{20} = -\tfrac{1}{2}$.

All the predictors of the type

$$y_{n+3} = A_0 y_{n+2} + A_1 y_{n+1} + A_2 y_n + h\,[B_0 f_{n+2} + B_1 f_{n+1} + B_2 f_n] + \frac{9 - A_1}{4}\, h^4 y^{(4)}(\xi) \qquad \ldots(3.8)$$

where $A_0 = 1 - A_1 - A_2$, $\quad B_0 = \dfrac{-16 + 8A_1 + 16A_2}{12}$

$$B_1 = \frac{23 + 5A_1 + 4A_2}{12}, \quad B_2 = \frac{5 + 4A_2 - A_1}{12}$$

are 3 step predictors of order 3 (Hamming 1962, p. 201). Substituting (3.7) and (3.8) in (2.14*a*), we get

$$[\{(9/64) - (3/4)(3/8)(-1/2 + A_1 - A_2)\}/(-1/2)(3/4)] = d$$

or $d = [3(D - 1)/4]$, where $D = A_1 - A_2$ $\qquad \ldots(3.9)$

As discussed in remark (3) of §2, our d should satisfy (2.15) and should be subject to the condition of low round off and truncation errors. In the present case as $\xi_{20} = -\tfrac{1}{2}$, (2.15) becomes

$$\ln(2) \geqslant dH$$

or $$0.7 \geqslant dH.$$

If we are solving problem (1.3) by the corrector (3.7), $H = \lambda h$, so that

$$0.7 \geqslant \lambda hd. \qquad \text{...(3.10)}$$

We now consider two predictors in which we fix $A_1 = 0$, so that they have same truncation errors viz, $(3/8)\, h^4 y^{(4)}(\xi)$.

(a) Let $D = -1$ so that $d = -3/2$ [from (3.9)] and $A_2 = A_1 - D = 1$. Now (3.8) becomes

$$y_{n+3} = y_n + (h/4)\,[9 f_{n+2} + 3 f_n] + (3/8)\, h^4 y^{(4)}\xi. \qquad \text{...(3.11)}$$

(b) Let $D = 2$ so that $d = 3/4$ [from (3.9)] and $A_2 = A_1 - D = -2$. Now (3.8) becomes

$$y_{n+3} = 3y_{n+2} - 2y_{n+1} + \frac{h}{12}\,(15 f_{n+2} - 48 f_{n+1} - 3 f_n)$$
$$+ \tfrac{3}{8} h^4 y^{(4)}(\xi). \qquad \text{...(3.12)}$$

From our discussions we see that for (3.7), the predictor (3.11) is better when $H > 0$ and (3.12) is a better predictor when $H < 0$. Table VI shows the results when $y' = y$; $y(0) = 1$ is solved by these two $(P - c)$ sets for $h = 0.2$ and 0.5. We note that the condition (3.10) which now becomes $d \leqslant (0.7)/h$ is satisfied. Table VII shows the results when $y' = -y$; $y(0) = 1$ is solved by these two $(P - c)$ sets with $h = 0.2$.

TABLE VI

$y' = y; \quad y(0) = 1$

x	Relative error $\times 10^{10}$		Relative error $\times 10^{6}$	
	(3.7, 3.11)	(3.7, 3.12)	(3.7, 3.11)	(3.7, 3.11)
	$h = 0.2$		$h = 0.5$	
2.0	189256	269093	1858	2309
4.0	416138	603210	5083	7396
6.0	643015	937361	8304	12485
8.0	869886	1271500	11514	17547
10.0	109675	1605628	14714	22584
12.0	1323614	1939745	17903	27595
14.0	1550470	2213550	21082	32580
16.0	1777321	2607144	24251	37540

TABLE VII

$y' = -y; \quad y(0) = 1$

x	(− Relative error × 10^8) (3.7, 3.11)	(3.7, 3.12)
	$h = 0.2$	
2.0	42785	39716
4.0	123430	85555
6.0	191166	131436
8.0	258939	160229
10.0	326756	201625
12.0	394620	243038
14.0	462530	284469
16.0	530485	325916

The condition (3.10), which is now $d \leqslant (0.7)/(-h)$ is satisfied. The results in both these examples are found to be in accordance with our discussions. The predictor (3.12) is not suitable for $y' = -y$; $y(0) = 1$ with $h = 0.5$, as in this case round off errors will be comparable to the solution as x increases.

4. NUMERICAL RESULTS

All the calculations are made on the computer system IBM 360/44 using double precision. The relative error in the solution y given by $\frac{y(x_n) - y_n}{y(x_n)}$, where $y(x_n)$ is the exact solution at $x = x_n$ is used to compare the methods.

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