ON THE UNIFORM NÖRLUND SUMMABILITY OF LEGENDRE SERIES

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Saxena (1965) and Dwivedi (1970) established theorems for uniform harmonic summability of Fourier series and uniform harmonic summability of Legendre series respectively. The present theorem deals with the uniform Nörlund summability of Legendre series which is more general than the previous two.

§1. Let

$$u_0(x) + u_1(x) + u_2(x) + \dots$$

be an infinite series, and

$$U_\nu(x) = u_0(x) + u_1(x) + \dots + u_\nu(x).$$

Let $\{q_n\}$ be a sequence of constants, real or complex and let

$$Q_n = q_0 + q_1 + q_2 + \dots + q_n.$$

We define sequence-to-sequence transformation

$$t_n(x) = \frac{1}{Q_n} \sum_{\nu=0}^{n} q_{n-\nu}\, U_\nu(x). \quad (Q_n \neq 0).$$

If $t_n(x) \to U(x)$ as $n \to \infty$ we write

$$\sum_{\nu=0}^{\infty} u_\nu(x) = U(x)\,(N, q_n)$$

or

$$U_\nu(x) \to U(x)\,(N, q_n).$$

If $\lim_{n\to\infty} [t_n(x) - U(x)] = o\,(1)$

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uniformly in a set E, then we say that the series $\Sigma\, u_n(x)$ is summable (N, q_n) uniformly in E to the sum $U(x)$.

§2. The Legendre series, associated with a Lebesgue integrable function in the interval defined by $-1 \leqslant x \leqslant 1$, is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) \qquad \ldots(2.1)$$

where

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{+1} f(x)\, P_n(x)\, dx$$

and the nth Legendre polynomial $P_n(x)$ is defined by the following expansion

$$\frac{1}{(1 - 2xz + z^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)\, z^n.$$

We use the following notations :

$$\psi(t) = \psi(\theta, t) = f\{\cos(\theta - t)\} - f(\cos\theta);$$

$$\Psi(t) = \int_0^t |\, \psi(u)\, |\, du$$

$$\tau = \left[\frac{1}{t}\right], \text{ where } [t] \text{ denotes the integral part of } t.$$

§3. Saxena (1965) established the following theorem for the uniform harmonic summability of a Fourier series.

Theorem A — If

$$\int_0^t |\, f(x+u) + f(x-u) - 2s\, |\, du = O\left(\frac{t}{\log \frac{1}{t}}\right)$$

uniformly in a set E in which $s = s(x)$ is bounded as $t \to +0$, then the Fourier series of a function $f(t)$, is summable by harmonic means uniformly in E to the sum s.

Dwivedi (1970) gave a corresponding result for uniform harmonic summability of Legendre series and proved the following theorem.

Theorem B — If

$$\int_0^t |f(x+u) - f(x)|\, du = O\left(\frac{t}{\log \frac{1}{t}}\right)$$

uniformly in a set E defined in the interval $(-1, +1)$, in which $f(x)$ is bounded, as $t \to +0$, then the series (2.1) is summable by harmonic means uniformly in E to the sum $f(x)$.

The object of the present paper is to generalize Theorem B for uniform Nörlund summability. We prove the following theorem.

Theorem — If

$$\int_0^t |f(x \pm u) - f(x)|\, du = o\left[\frac{\lambda(1/t)\, q_\tau}{\beta\,(Q_\tau)}\right] \qquad \ldots(3.1)$$

uniformly in a set E defined in the interval $(-1, +1)$, in which $f(x)$ is bounded, as $t \to +0$, then the series (2.1) is summable (N, q_n) uniformly in E to the sum $f(x)$, where $\lambda(t)$ and $\beta(t)$ are functions of t such that $\lambda(t)$, $\beta(t)$ and $\frac{\lambda(t)t}{\beta(t)}$ increase monotonically with t as $t \to +0$ and

$$\lambda(n)\,.\,Q_n = o\,[\beta(Q_n)] \text{ as } n \to \infty$$

and $\{q_n\}$ is a sequence of real non-negative and non-increasing constants, such that Q_n tends to infinity with n.

§4. Following lemmas are required for the proof of our theorem.

Lemma 1 (Sansone 1959)

$$\sum_{\nu=1}^{n} (2\nu+1)\, P_\nu(x)\, P_\nu(y) = \frac{(n+1)\,\{P_{n+1}(y)\, P_n(x) - P_n(y)\, P_{n+1}(x)\}}{y - x}. \qquad \ldots(4.1)$$

This identify it as Christoffel's formula of summation.

Lemma 2 — Under the condition (3.1), we have

$$\int_0^t |f\{\cos(\theta - \nu)\} - f(\cos\theta)|\, d_\nu = o\left[\frac{\lambda(1/t)\, q_\tau}{\beta(Q_\tau)}\right] \qquad \ldots(4.2)$$

as $t \to +0$, where $x = \cos\theta$, $x + u = \cos\phi$, and $\theta - \phi = \nu$.

The proof of this lemma follows on the line of Foa (1943).

Lemma 3 (*McFadden* 1942) — For $0 \leqslant a < b \leqslant \infty$, $0 < t < \pi$ and any n,

$$\left| \sum_{k=a}^{b} q_n \exp\left(i(n-k)t\right) \right| < A\, Q_\tau \qquad \ldots(4.3)$$

where A is an absolute constant.

§5. *Proof of the Theorem* — The nth partial sum of the series (2.1) is

$$S_n(x) = \sum_{\nu=0}^{n} a_\nu\, P_\nu(x)$$

$$= \frac{n+1}{2} \int_{-1}^{+1} f(y)\, \frac{P_{n+1}(y)\, P_n(x) - P_n(y)\, P_{n+1}(x)}{y-x}$$

by Lemma 1.

Putting $f(y) \equiv 1$, it can be easily seen that

$$1 = \frac{n+1}{2} \int_{-1}^{+1} \frac{P_{n+1}(y)\, P_n(x) - P_n(y)\, P_{n+1}(x)}{y-x}\, dy.$$

Therefore,

$$S_n(x) - f(x) = \frac{n+1}{2} \int_{-1}^{+1} [\, f(y) - f(x)\,]\, \frac{P_{n+1}(y)\, P_n(x) - P_n(y)\, P_{n+1}(x)}{y-x}\, dy$$

and so

$$S_{n-k}(x) - f(x) = \frac{n-k+1}{2} \int_{-1}^{+1} [\, f(y) - f(x)]$$

$$\times \frac{P_{n-k+1}(y)\, P_{n-k}(x) - P_{n-k}(y)\, P_{n-k+1}(x)}{y-x}\, dy.$$

Let us take a positive number S, less than 1 and consider it as the sum of two other positive numbers μ and δ. Let d be another positive number, such that $0 < d < \mu$ and μ_x and $\mu_{x'}$ be two continuous functions of x within $(-1, +1)$, which lie within the limits

$$d \leqslant \mu_x \leqslant \mu\,,\; d \leqslant \mu_{x'} \leqslant \mu.$$

Therefore, for $-1+S \leqslant x \leqslant 1-S$, we have

$$S_{n-k}(x) - f(x) = \frac{n-k+1}{2}\left[\int_{-1}^{x-\mu_x} + \int_{x-\mu_x}^{x+\mu_x'} + \int_{x+\mu_x'}^{+1}\right][f(y)-f(x)]$$

$$\times \frac{P_{n-k+1}(y)\, P_{n-k}(x) - P_{n-k}(y)\, P_{n-k+1}(x)}{y-x}\, dy$$

$$= A_{n-k}(x) + B_{n-k}(x) + C_{n-k}(x), \text{ say.} \qquad \ldots(5.1)$$

Hobson (1909) has shown that uniformly for $-1+S \leqslant x \leqslant 1-S$,

$$\lim_{n\to\infty} A_{n-k}(x) = 0 \text{ and } \lim_{n\to\infty} C_{n-k}(x) = 0 \qquad \ldots(5.2)$$

uniformly in the set E.

Now we suppose $x = \cos\theta$, $y = \cos\phi$, $0 \leqslant \theta < \pi$, $0 < \phi < \pi$, $1-\delta = \cos\rho$, $1-(\mu+\delta) = 1-S = \cos(\rho+\delta)$, $0 < \rho < \pi/2$, $0 < \sigma$; $\rho+\sigma < \pi/2$.
Thus if η denotes the minimum of

$$[\text{arc} \cos u - \text{arc}\,(u+\mu)]$$

for u in $(-1, 1-\mu)$, we have on the lines of Sansone (1959)

$$B_{n-k}(\cos\theta) = \frac{n-k+1}{2}\int_{\theta-\eta}^{\theta+\eta} [f(\cos\phi) - f(\cos\theta)]$$

$$\times \frac{P_{n-k+1}(\cos\phi)\, P_{n-k}(\cos\theta) - P_{n-k}(\cos\phi)\, P_{n-k+1}(\cos\theta)}{\cos\phi - \cos\theta} \sin\phi\, d\phi$$

in which $\rho+\sigma \leqslant \theta \leqslant \pi-(\rho+\sigma)$; $0 < \eta \leqslant \sigma$.
With successive transformation, we get

$$B_{n-k}(\cos\theta) = D_{n-k}(\theta) + E_{n-k}(\theta), \text{ say.} \qquad \ldots(5.3)$$

where

$$D_{n-k}(\theta) = \frac{1}{2\pi \sin^{1/2}\theta}\int_{\theta-\eta}^{\theta+\eta} \frac{f(\cos\phi)-f(\cos\theta)}{\sin\frac{1}{2}(\theta-\phi)} \sin(n-k+1)(\theta-\phi) \sin^{1/2}\phi\, d\phi$$

and obviously on the line of Sansone (1959) $E_{n-k}(\theta) = o(1)$, as $n \to \infty$ uniformly where x lies within $(-1+S, 1-S)$ i.e., in the set E.

Putting $(\theta-\phi) = t$, we get

$$D_{n-k}(\theta) = \frac{1}{\pi \sin^{1/2}\theta}\int_0^{\eta} \frac{f\{\cos(\theta-t)\} - f(\cos\theta)}{\sin\frac{1}{2}t}$$

$$\times \sin(n-k+1)t \sin^{1/2}(\theta-t)\, dt. \qquad \ldots(5.4)$$

So, we have from (5.1) to (5.4)

$$S_{n-k}(x) - f(x) = \frac{1}{\pi \sin^{1/2}\theta} \int_0^{\eta} \frac{f\{\cos(\theta - t)\} - f(\cos\theta)}{\sin\frac{1}{2}t}$$

$$\times \sin(n-k+1)\,t \sin^{1/2}(\theta - t)\, dt + o(1).$$

Now

$$\frac{1}{Q_n}\sum_{k=0}^{n} q_k\{S_{n-k}(x) - f(x)\} = \frac{1}{Q_n}\sum_{k=0}^{n} q_k \frac{1}{\pi \sin^{1/2}\theta}$$

$$\times \int_0^{\eta} \frac{f\{\cos(\theta - t)\} - f(\cos\theta)}{\sin\frac{1}{2}t} \sin(n-k+1)\,t \sin^{1/2}(\theta - t)\, dt + o(1)$$

uniformly in E

$$= \frac{1}{\pi \sin^{1/2}\theta} \int_0^{\eta} [f\{\cos(\theta - t)\} - f(\cos\theta)] \sin^{1/2}(\phi - t) \frac{1}{Q_n}$$

$$\times \sum_{k=0}^{n} q_k \frac{\sin(n-k+1)\,t}{\sin\frac{1}{2}t}\, dt + o(1) \text{ uniformly in } E$$

$$= O\left[\int_0^{\eta} |\psi(t)| \,.\, |N_n(t)|\, dt\right] + o(1) \text{ uniformly in } E$$

$$= O\left[\int_0^{1/n} |\psi(t)| \,.\, |N_n(t)|\, dt\right]$$

$$+ O\left[\int_{1/n}^{\eta} |\psi(t)| \,.\, |N_n(t)|\, dt\right] + o(1) \text{ uniformly in } E$$

$$= I_1 + I_2 + o(1), \text{ say}$$

where

$$N_n(t) = \frac{1}{Q_n}\sum_{k=0}^{n} q_k \frac{\sin(n-k+1)\,t}{\sin\frac{1}{2}t}\, dt.$$

In order to prove the theorem we have to show that under assumptions of the theorem

$$I_1 = o(1) \text{ and } I_2 = o(1) \text{ as } n \to \infty \text{ uniformly in } E. \qquad \ldots(5.5)$$

Now uniformly in $0 < t \leqslant \frac{1}{n}$

$$N_n(t) = O(n).$$

So

$$I_1 = O\,[\int_0^{1/n} |\,\psi(t)\,|\,.\,|\,N_n(t)\,|\,dt]$$

$$= O\,[\,n \int_0^{1/n} |\,\psi(t)\,|\,dt]$$

$$= o\left[\,n\,\frac{\lambda(n)\,q_n}{\beta(Q_n)}\right]$$

$$= o\,(1), \text{ as } n \to \infty, \text{ uniformly in } E, \text{ since } n\,q_n \leqslant Q_n. \qquad \ldots(5.6)$$

Now uniformly for $n^{-1} \leqslant t \leqslant \eta$ we have

$$N_n(t) = O\left[\frac{Q_\tau}{tQ_n}\right]$$

So

$$I_2 = O\,[\int_{1/n}^{\eta} |\,\psi(t)\,|\,.\,|\,N_n(t)\,|\,dt]$$

$$= O\left[\frac{1}{Q_n}\left\{\int_{1/n}^{\eta} |\,\psi(t)\,|\,.\,\frac{Q_\tau}{t}\right\} dt\right.$$

$$= O\left[\frac{1}{Q_n}\left\{\,\Psi(t)\,\frac{Q_\tau}{t}\right\}_{1/n}^{\eta}\,\right]$$

$$\times\, O\left[\,\frac{1}{Q_n}\int_{1/n}^{\eta} \Psi(t)\,\frac{Q_\tau}{t^2}\,dt\,\right]$$

$$+\, O\left[\frac{1}{Q_n}\int_{1/n}^{\eta} \Psi(t)\,\frac{1}{t}\,dQ_\tau\,\right]$$

$$= o\left(\frac{1}{Q_n}\right) + o\left(\frac{nq_n}{Q_n}\right) + o\left[\frac{1}{Q_n}\int_{1/n}^{\eta}\frac{\lambda(1/t)q_\tau\,Q_\tau}{\beta(Q_\tau)\,t^2}\,dt\right]$$

$$+\, o\left[\frac{1}{Q_n}\int_{1/n}^{\eta}\frac{\lambda(1/t)q_\tau}{\beta(Q_\tau)}\,\frac{1}{t}\,|\,dQ_\tau\,|\,\right]$$

$$= o(1) + o\left[\frac{1}{Q_n}\int_{1/\eta}^{n} q_{[s]}\,ds\right] + o\left[\frac{1}{Q_n}\int_{1/\eta}^{n} dQ_{[s]}\right]$$

$$= o(1) + o\left[\frac{1}{Q_n}\sum_{\nu=0}^{n} q_\nu\right]$$

$$= o(1), \text{ uniformly in } E. \qquad \ldots(5.7)$$

So from (5.6) and (5.7) we get (5.5) and this completes the proof of the theorem.

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References

Dwivedi, G. K. (1970). Absolute Nörlund summability of factored Fourier series. Under publication in *Bulletin* 1 *Academic Polonaise Des Sciences* (*Poland*).

Foa, A. (1943). Sulla Sommabilita forte della Serie di Legendre. *Boll. Univ. Mat. Ital.* (2), **5**, 18-27.

Hobson, E. W. (1909). On the representation of a function by a series of Legendre's functions. *Proc. Lond. math. Soc.* (2), **7**, 24-29.

McFadden, L. (1942). Absolute Nörlund summability. *Duke math. J.*, **9**, 168-207.

Sansone, G. (1959). Orthogonal Functions. English Edition. p. 179, pp. 227-33.

Saxena, A. (1965). On uniform harmonic summability of Fourier series and its conjugate series. *Proc. natn. Inst. Sci. India*, **31**, 303-10.