THE FINE SPECTRA OF THE HÖLDER SUMMABILITY OPERATORS

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Let C_1^p (p a positive integer) denote the classical Hölder summability method of order p. The method C_1^p is conservative and hence defines a continuous linear operator on c, the Banach space of convergent sequences of complex numbers. In this paper the fine spectra of each of these operators is presented, where the operator is viewed as a member of the Banach space of continuous linear operators on c. In deriving the fine spectra a state diagram is employed. Applications to summability theory are considered.

1. INTRODUCTION

Let C_1 denote the classical Cèsaro summability method or the Hölder method of order 1. Because C_1 is a conservative summability method, it defines a continuous linear operator on c, the Banach space of convergent sequences of complex numbers. In addition, C_1^p (p a positive integer), which is the Hölder method of order p, is also conservative and defines a continuous linear operator on c.

The main purpose of this paper is to present the fine spectrum of each of these operators where the operator is viewed as a member of $B[c]$, the Banach space of continuous linear operators on c. The fine spectra of C_1 and C_1^P are developed in Sections 3 and 4, respectively. In Section 5 an application to summability theory is considered.

2. BACKGROUND AND NOTATION

The matrix $T = (t_{nk})$, $n, k = 0, 1, 2\ldots$, is said to sum the sequence $x = \{x_n\}$, $n = 0, 1, 2,\ldots$ if the sequence $Tx = \{(Tx)_n\}$, where $(Tx)_n = \sum_k a_{nk}x_k$ is convergent. The convergence domain of a matrix is denoted by $c_T = \{x : Tx \in c\}$. If $c_T \supset c$, T is called conservative and if $\lim Tx = \lim x$ for all $x \in c$, T is called regular. A matrix $T = (t_{nk})$ is conservative if and only if

(i) $\lim_{n} t_{nk} = \alpha_k, k = 0, 1, 2,\ldots,$

(ii) $\lim_{n} \sum_{k=0}^{\infty} t_{nk} = \zeta,$

(iii) $\sup_{n} \sum_{k=0}^{\infty} | t_{nk} | < \infty.$

A matrix T is regular if and only if $\alpha_k=0, k=0, 1, 2,\ldots$ in (i), $\zeta=1$ in (ii), and (iii) holds. The norm of the matrix T, whether conservative or not, is $\|T\| = \sup_{n} \sum_{k} | t_{nk} |$. If T is conservative, T defines a continuous linear operator on c and $\|T\|$ as a member of $B[c]$ is the same as $\|T\|$ as a matrix. A matrix $T = (t_{nk})$ is called a triangle if $t_{nk} = 0$ when $k > n$ and $a_{kk} \neq 0$.

The conjugate of c is congruent to l_1, the Banach space of sequences of complex numbers $x = \{x_n\}$ such that $\sum_{n} | x_n | < \infty$, with $\|x\| = \sum_{n} | x_n |$. Thus T', the conjugate operator of T, is an operator on l_1. If $U \in B[l_1]$, then U is given by a matrix and the matrix $U = (u_{nk})$ is a member of $B[l_1]$ if and only if $\sum_{n} | u_{nk} | \leqslant m$ (m independent of k). If $U \in B[l_1]$, then $\|U\| = \sup_{k} \sum_{n} | u_{nk} |$. If T is a conservative matrix operator on c, $T' = (t'_{nk})$ is defined by $t_{00} = \lim T1 - \sum_{n} (\lim T\delta^n)$; $t_{0k} = 0$, $k = 1, 2, 3,\ldots$; $t_{n0} = \lim T\delta^{n-1}$; $n = 1, 2, 3,\ldots$; and $t_{nk} = (T\delta^{n-1})_{k-1}$, n, k otherwise; where δ^n is the sequence having all the coordinates 0, except the nth coodinate, which is 1.

To obtain the fine spectra of the operators $T(\alpha)$ studied in this paper, we use the state diagram displayed in Figure 1 which is found in (Goldberg 1966, p. 66). For this purpose we use the following notation which is also found in (Goldberg 1966). If $T \in B[X]$, where X is a Banach space, we list three possibilities for $R(T)$, the range of T:

(I) $R(T) = X,$

(II) $\overline{R(T)} = X$, but $R(T) \neq X$

(III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways nine different states are created. These are labelled by : $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3$. If an operator is in state III_2, for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous. Usually we use the notation $T \in III_2$ instead of writing "T is in state III_2." Similarly we write $T \in 2$ to indicate that T^{-1} exists and is discontinuous or $T \in III$ when $\overline{R(T)} \neq X$.

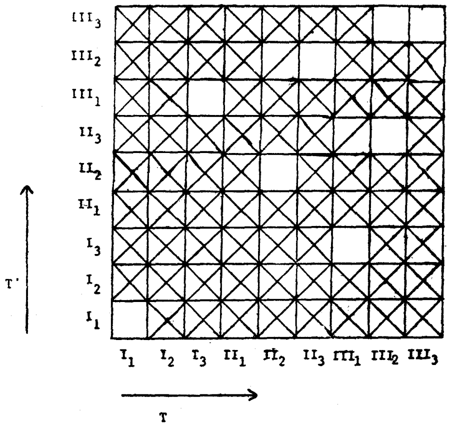

FIG. 1. State diagram for $B[X]$ and $B[X']$ for a non-reflective Banach space X.

If λ, a complex number, is such that $\lambda I - T \in I_1$ or II_1, then λ is in the resolvent set of T. This set of values is denoted by $\rho(T)$. All scalar values of λ not in $\rho(T)$ comprise the spectrum of T, which is denoted by $\sigma(T)$. The further classification of $\sigma(T)$ gives rise to the fine spectrum of T. If $\lambda I - T$ is in a given state, III_1 (say), then we write $\lambda \in III_1\sigma(T)$.

3. THE FINE SPECTRUM OF C_1

In this section we develop the fine spectrum of the matrix operator $C_1 = (c_{nk})$, where $c_{nk} = \dfrac{1}{n+1}$, $k \leqslant n$ and $c_{nk} = 0, n, k$ otherwise. Hence $\lambda I - C_1 = (c_{nk})$ where $c_{nk} = \lambda - \dfrac{1}{n+1}$, $k = n$; $c_{nk} = -\dfrac{1}{n+1}$, $k < n$; and $c_{nk} = 0, n, k$ otherwise and using the matrix for T' in section 2 we have $\lambda I - C_1' = (c'_{nk})$, where $c'_{00} = \lambda - 1$; $c'_{nk} = 0, k < n$; $c'_{kk} = \lambda - \dfrac{1}{k}$, $k \geqslant 1$; $c'_{nk} = -\dfrac{1}{k}$, $k > n$; and $c'_{0k} = 0, k \geqslant 1$. It should be noted that $(\lambda I - C_1)' = \lambda I - C_1'$ and therefore $\lambda I - C_1'$ is the conjugate of $\lambda I - C_1$.

Since $\|C_1\|$ is clearly 1, $\lambda \in \rho(C_1)$ if $|\lambda| > 1$. In the first theorem of this section an enlargement of $\rho(C_1)$ is obtained.

Theorem 1 — If $\operatorname{Re}\frac{1}{\lambda} < 1$, then $\lambda \in \rho(C_1)$.

PROOF: Let λ be a complex number such that $\operatorname{Re}\frac{1}{\lambda} < 1$. Since $\lambda \neq 1, 1/2, 1/3, \ldots, \lambda I - C_1$ is a triangle and therefore one-to-one.

In the next part of the proof it is established that $\lambda I - C_1$ is onto. If $y = (y_0, y_1, y_2, \ldots)$ is an arbitrary element in c, a few calculations reveal that the sequence $x = (x_0, x_1, x_2, \ldots)$, where

$$x_0 = \frac{1}{\lambda - 1} y_0$$

$$x_n = \frac{1}{\lambda - \frac{1}{n+1}} y_n + \frac{1}{(n+1)\left(\lambda - \frac{1}{n+1}\right)(\lambda - 1/n)} y_{n-1}$$

$$+ \frac{\lambda}{(n+1)\left(\lambda - \frac{1}{n+1}\right)\left(\lambda - \frac{1}{n}\right)\left(\lambda - \frac{1}{n-1}\right)} y_{n-2}$$

$$+ \ldots + \frac{\lambda^{n-1}}{(n+1)\left(\lambda - \frac{1}{n+1}\right)\left(\lambda - \frac{1}{n}\right)\ldots(\lambda - 1)} y_0$$

$n = 1, 2, 3, \ldots$, is mapped into y by $\lambda I - C_1$. Hence there is a sequence x which is carried into y by $\lambda I - C_1$, but we must establish that $x \in c$.

Now, the above equations define a matrix transformation where the matrix (a_{nk}) is given by $a_{nk} = \dfrac{1}{\lambda - \frac{1}{n+1}}$, $k = n$;

$$a_{nk} = \frac{\lambda^{n-(k+1)}}{(n+1)\left(\lambda - \frac{1}{k+1}\right)\left(\lambda - \frac{1}{k+2}\right)\cdots\left(\lambda - \frac{1}{n+1}\right)},$$ $k < n$; and $a_{nk} = 0$, n, k otherwise. For $n = 1, 2, 3, \ldots$, a series of calculations reveal that

$$\sum_{k=0}^{\infty} a_{nk} = \frac{1}{\lambda - 1}. \text{ It is trivial that } \sum_{k=0}^{\infty} a_{0k} = \frac{1}{\lambda - 1}.$$

Also,

$$\sum_{k=0}^{\infty} | a_{nk} | = \frac{1}{|\lambda|^2} \left[\frac{1}{(n+1)\left|1 - \frac{1}{\lambda}\right| \cdots \left|1 - \frac{1}{(n+1)\lambda}\right|} + \right.$$

(*equation continued on p.* 699)

$$+\frac{1}{(n+1)\left|1-\frac{1}{2\lambda}\right|\dots\left|1-\frac{1}{(n+1)\lambda}\right|}$$

$$\left.+\frac{1}{(n+1)\left|1-\frac{1}{n\lambda}\right|\left|1-\frac{1}{(n+1)\lambda}\right|}\right]+\frac{1}{\left|\lambda-\frac{1}{n+1}\right|}.$$

Now $\left|1-\frac{1}{\lambda}\right| \geqslant \operatorname{Re}\left(1-\frac{1}{\lambda}\right)=1-\operatorname{Re}\frac{1}{\lambda}>0$. Let $\beta=1-\operatorname{Re}\frac{1}{\lambda}$. Then

$$\left|1-\frac{1}{n\lambda}\right| \geqslant \frac{\beta+(n-1)}{n}, \; n=1,2,3,\dots .$$

Using this inequality several detailed calculations reveal that for $n = 1, 2, 3, \dots$,

$$\sum_{k=0}^{\infty}|a_{nk}| \leqslant \frac{1}{|\lambda|^2}\frac{1}{\beta}\frac{1}{\frac{\beta}{n}+1}+\frac{1}{\left|\lambda-\frac{1}{n+1}\right|} \to \frac{1}{n|\lambda|^2}\frac{1}{\beta}+\frac{1}{|\lambda|}.$$

Hence $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$.

For fixed k and for $n = k+1, k+2, \dots$,

$$a_{nk} = \frac{\lambda^{n-(k+1)}}{(n+1)\left(\lambda-\frac{1}{k+1}\right)\left(\lambda-\frac{1}{k+2}\right)\dots\left(\lambda-\frac{1}{n+1}\right)}$$

$$=\frac{(\lambda-1)\left(\lambda-\frac{1}{2}\right)\dots\left(\lambda-\frac{1}{k}\right)}{\lambda^{k+2}}$$

$$\times\frac{(-1)^{n+1}n!\,n^{-(1/\lambda)+1}}{\left(-\frac{1}{\lambda}+1\right)\left(-\frac{1}{\lambda}+2\right)\dots\left(-\frac{1}{\lambda}+1+n\right)}\frac{1}{n^{-1/\lambda+1}}$$

Hence

$$\lim_n a_{nk} = \frac{(\lambda-1)\left(\lambda-\frac{1}{2}\right)\dots\left(\lambda-\frac{1}{k}\right)}{\lambda^{k+2}}$$

$$\times \lim_n \frac{n!\,n^{-(1/\lambda)+1}}{\left(-\frac{1}{\lambda}+1\right)\left(-\frac{1}{\lambda}+2\right)\dots\left(-\frac{1}{\lambda}+1+n\right)}\frac{(-1)^{n+1}}{n^{-(1/\lambda)+1}}$$

$$=\frac{(\lambda-1)\left(\lambda-\frac{1}{2}\right)\dots\left(\lambda-\frac{1}{k}\right)}{\lambda^{k+2}}\Gamma\left(-\frac{1}{\lambda}+1\right).0=0$$

where Γ is the classical gamma function. We remark that this limit is zero because $\text{Re}\left(-\frac{1}{\lambda}+1\right) > 0$.

The conditions stated in Section 2 hold and it follows that the matrix is conservative. This means that $x \in c$, therefore $\lambda I - C_1 \in I$. Since the space c is complete, state I_2 is impossible, and upon consulting the state diagram, we find that $\lambda I - C_1 \in I_1$, i.e., $\lambda \in \rho(C_1)$.

Our next result is the following : if Re $1/\lambda > 1$, then $\lambda \in III_1\sigma(C_1)$. Since a separate argument is needed when $\lambda = \frac{1}{j}$, $j = 2, 3, 4, \ldots$, this result is contained in the next two theorems.

Theorem 2 — If $\text{Re}\frac{1}{\lambda} > 1$, $\lambda \neq \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, then $\lambda \in III_1\sigma(C_1)$.

PROOF : The matrix $\lambda I - C_1$ is a triangle and therefore as an operator $\lambda I - C_1$ is one-to-one.

Now consider the conjugate operator $\lambda I - C_1'$. If the $(\lambda I - C_1')\, x = 0$, then $(\lambda - 1)\, x_0 = 0$ and

$$\left(\lambda - \frac{1}{n}\right) x_n - \sum_{i=n+1}^{\infty} \frac{x_i}{i} = 0,\ n = 1, 2, 3, \ldots$$

These equations imply $x_0 = 0$ and

$$x_n = \frac{(\lambda - 1)\left(\lambda - \frac{1}{2}\right) \cdots \left(\lambda - \frac{1}{n-1}\right)}{\lambda^{n-1}}\, x_1,\ n = 2, 3, 4, \ldots .$$

Rewriting the latter equation we have, for $n = 2, 3, 4, \ldots$,

$$x_n = \frac{\left(\frac{1}{\lambda} - 1\right)\left(\frac{1}{\lambda} - 2\right) \cdots \left(\frac{1}{\lambda} - (n-1)\right)}{(n-1)!} (-1)^{n-1} x_1$$

$$= \binom{\frac{1}{\lambda} - 1}{n - 1} (-1)^{n-1} x_1.$$

Now

$$\sum_{n=2}^{\infty} \binom{\frac{1}{\lambda} - 1}{n - 1} (-1)^{n-1} x_1 = x_1 \sum_{n=1}^{\infty} \binom{\frac{1}{\lambda} - 1}{n} (-1)^n$$

coverges absolutely when $\operatorname{Re}\left(\frac{1}{\lambda} - 1\right) > 0$. Hence x_1 need not be zero for x to be in l_1 and $\lambda I - C_1'$ is not one-to-one. Consulting the state diagram we see $\lambda I - C_1 \in III_1 \cup III_2$.

To prove that $\lambda I - C_1 \in III_1$ it is sufficient to show that $\lambda I - C_1'$ is onto. To this end, equate

$$\left((\lambda - 1)\, x_0, (\lambda - 1)\, x_1, -\sum_{i=2}^{\infty} \frac{x_i}{i}, \left(\lambda - \frac{1}{2}\right) x_2 - \sum_{i=3}^{\infty} \frac{x_i}{i}, \ldots\right)$$
$$= (y_0, y_1, y_2, \ldots)$$

where $y = (y_0, y_1, y_2, \ldots)$ is an arbitrary element of l_1. The above equation holds only if

$$(\lambda - 1)\, x_0 = y_0,$$

$$\left(\lambda - \frac{1}{n}\right) x_n - \sum_{i=n+1}^{\infty} \frac{x_i}{i} = y_n, \quad n = 1, 2, 3, \ldots .$$

If we choose $x_1 = 0$ and solve for the remaining x's in terms of the y's, we obtain

$$x_0 = \frac{1}{\lambda - 1}\, y_0$$

$$x_2 = \frac{1}{\lambda}\,[y_2 - y_1]$$

$$\vdots$$

$$x_n = \frac{1}{\lambda}\left[y_n - \frac{1}{(n-1)\lambda}\, y_{n-1} - \frac{1}{(n-2)\lambda}\left(1 - \frac{1}{(n-1)\lambda}\right) y_{n-2}\right.$$
$$- \cdots - \frac{1}{2\lambda}\left(1 - \frac{1}{(n-1)\lambda}\right)\left(1 - \frac{1}{(n-2)\lambda}\right) \cdots \left(1 - \frac{1}{3\lambda}\right) y_2$$
$$\left. - \left(1 - \frac{1}{(n-1)\lambda}\right) \cdots \left(1 - \frac{1}{2\lambda}\right) y_1\right],$$

$n = 3, 4, 5, \ldots$. These equations define a matrix transformation. Denote this matrix by $A = (a_{nk})$.

By hypothesis $\operatorname{Re}\left(\frac{1}{\lambda} - 1\right) > 0$ and hence there exists a positive real number β such that $0 < \beta < \operatorname{Re}\left(\frac{1}{\lambda} - 1\right)$. Furthermore, there exists a positive integer K such that

$$\left| 1 - \frac{\frac{1}{\lambda} - 1}{k} \right| < 1 - \frac{\beta}{k}$$

for all $k > K$. Using this inequality detailed calculations which can easily be carried out will show that for fixed $k > \max(1, K)$

$$\sum_{n=0}^{\infty} | a_{nk} | = \left|\frac{1}{\lambda}\right| + \left|\frac{1}{\lambda}\right|^2 \frac{2 \cdot 3 \ldots (k-1)}{\beta(\beta-1) \ldots (\beta-(k-1))} \left[(-1)^k \sum_{j=0}^{\infty} \binom{\beta}{j} \right.$$

$$\left. \times (-1)^j - (-1)^k \sum_{j=0}^{k-1} \binom{\beta}{j} (-1)^j \right].$$

The infinite series in this equation is the well-known binomial series and since β is positive it converges. In addition, its sum is zero (Knopp 1947, p. 426). Therefore,

$$\sum_{n=0}^{\infty} | a_{nk} | < \left|\frac{1}{\lambda}\right| + \left|\frac{1}{\lambda}\right|^2 \cdot \frac{1}{\beta}$$

Let $M = \min \left(\left|\frac{1}{\lambda}\right| + \left|\frac{1}{\lambda}\right|^2 \cdot \frac{1}{\beta}, \sum_{n=0}^{\infty} | a_{n0} |, \ldots, \sum_{n=0}^{\infty} | a_{nk} | \right)$.

Then $\sum_{n=0}^{\infty} | a_{nk} | \leqslant M$, independent of k, and by the condition given in Section 2, the above matrix defines a mapping from l_1 into l_1, i.e., the sequence x is in l_1. Therefore, $\lambda I - C_1'$ is onto and $\lambda I - C_1 \in III_1$.

Our next theorem provides the states of the operators $\frac{1}{j} I - C_1, j = 2, 3, 4, \ldots$.

Theorem 3 — If $\lambda = \frac{1}{j}$, $j = 2, 3, 4, \ldots$, then $\lambda \in III_1\sigma(C_1)$.

PROOF : Let j be a fixed integer greater than or equal to 2. Now a sequence $x = (x_0, x_1, x_2, \ldots)$ is carried into

$$\left(\left(\frac{1}{j} - 1\right) x_0, -\frac{1}{2} x_0 + \left(\frac{1}{j} - \frac{1}{2}\right) x_1, \ldots, -\frac{1}{j} x_0 - \frac{1}{j} x_1 - \ldots \right.$$

$$- \frac{1}{j} x_{j-2}, -\frac{1}{j+1} x_0 - \frac{1}{j+1} x_1 - \ldots - \frac{1}{j+1} x_{j-1}$$

$$\left. + \left(\frac{1}{j} - \frac{1}{j+1}\right) x_j, \ldots \right)$$

by $\frac{1}{j}I - C_1$. From this it follows that $\left(\frac{1}{j}I - C_1\right)x = 0$ implies $0 = x_0 = x_1 = \dots = x_{j-2}$. A few calculations also reveal

$$x_{j+m} = \frac{j(j+1)\dots(j+m)}{(m+1)!}x_{j-1},\ m = 0, 1, 2, \dots .$$

Now

$$\lim_m x_{j+m} = x_{j-1}\lim_m \frac{j(j+1)\dots(j+m)}{(m+1)!}$$

$$= \frac{x_{j-1}}{(j-1)!}\lim_m (m+j)(m+j-1)\dots(m+2).$$

Hence $x \in c$ if and only if $x_{j-1} = 0$ which gives at once that $x = 0$. From this we conclude $\frac{1}{j}I - C_1 \in 1 \cup 2$.

Next we turn to the conjugate operator $\frac{1}{j}I - C_1'$. The sequence $x = (x_0, x_1, x_2, \dots) \in l_1$ is taken into

$$\Bigg(\left(\frac{1}{j}-1\right)x_0, \left(\frac{1}{j}-1\right)x_1 - \sum_{i=2}^{\infty}\frac{x_i}{i}, \left(\frac{1}{j}-\frac{1}{2}\right)x_2 - \sum_{i=2}^{\infty}\frac{x_i}{i}, \dots,$$

$$\left(\frac{1}{j}-\frac{1}{j-1}\right)x_{j-1} - \sum_{i=j}^{\infty}\frac{x_i}{i}, -\sum_{i=j+1}^{\infty}\frac{x_i}{i},$$

$$\left(\frac{1}{j}-\frac{1}{j+1}\right) - \sum_{i=j+2}^{\infty}\frac{x_i}{i}, \dots\Bigg)$$

by $\frac{1}{j}I - C_1'$. A brief examination of the image sequence reveals that non-zero sequences are mapped into the zero sequence. For, if $0 = x_{j+1} = x_{j+2} = \dots$, then the conditions

$$\left(\frac{1}{j}-1\right)x_1 - \frac{x_2}{2} - \dots - \frac{x_j}{j} = 0$$

$$\vdots$$

$$\left(\frac{1}{j}-\frac{1}{j-1}\right)x_{j-1} - \frac{x_j}{j} = 0$$

together with $x_0 = 0$ will insure that the sequence x will be mapped into zero. There exists a non-trivial solution to the above conditions because $x_1, \dots, x_j$ satisfy a linear homogeneous system consisting of $j-1$ equations in j unknowns. Thus,

$$\frac{1}{j} I - C_1' \in 3.$$

To obtain the conclusion it is sufficient to prove that $\frac{1}{j} I - C_1'$ is onto. Accordingly, let $y = (y_0, y_1, y_2, \ldots)$ be arbitrary in l_1.

If $x = (x_0, x_1, x_2, \ldots)$ exists such that $\left(\frac{1}{j} I - C_1\right) x = y$, we must have

$$\left(\frac{1}{j} - 1\right) x_0 = y_0$$

$$\left(\frac{1}{j} - 1\right) x_1 - \sum_{i=2}^{\infty} \frac{x_i}{i} = y_1$$

$$\vdots$$

$$\left(\frac{1}{j} - \frac{1}{j-1}\right) x_{j-1} - \sum_{i=j}^{\infty} \frac{x_i}{i} = y_{j-1}$$

$$- \sum_{i=j+1}^{\infty} \frac{x_i}{i} = y_j$$

$$\left(\frac{1}{j} - \frac{1}{j+1}\right) x_{j+1} - \sum_{i=j+2}^{\infty} \frac{x_i}{i} = y_{j+1}$$

$$\vdots$$

Choose $x_j = 0$. Then the above conditions enable one to solve for $x_1, x_2, \ldots, x_{j-1}$ in terms of $y_1, y_2, \ldots, y_j$. From these equations we also obtain the relations

$$x_{j+1} = j(y_{j+1} - y_j)$$

$$x_{j+2} = j\left(y_{j+2} - \frac{1}{j+1} y_{j+1} - \frac{1}{j+1} y_j\right)$$

$$\vdots$$

$$y_{j+m} = j\left(y_{j+m} - \frac{j}{j+m-1} y_{j+m-1} - \frac{m-1}{j+m-1} \frac{j}{j+m-2} y_{j+m-2}\right.$$

$$- \frac{m-1}{j+m-1} \frac{m-2}{j+m-2} \frac{j}{j+m-3} y_{j+m-3} -$$

(*equation continued on p.* 705)

$$-\frac{m-1}{j+m-1}\frac{m-2}{j+m-2}\frac{m-3}{j+m-3}\frac{j}{j+m-4}y_{j+m-4}$$

$$-\dots-\frac{m-1}{j+m-1}\frac{m-2}{j+m-2}\cdots\frac{1}{j+1}y_j\Big)$$

$$\vdots$$

These equations define a matrix transformation. Denote the matrix by $A = (a_{nk})$. Then

$$\sum_{n=0}^{\infty} | a_{n,j+m} | = j + \frac{j^2}{j+m}\left[1+\frac{m+1}{j+m+1}+\frac{(m+1)(m+2)}{(j+m+1)(j+m+2)}+\dots\right],$$

$$m = 1, 2, 3, \dots .$$

The infinite series in this expression can be summed. For,

$$1+\frac{m+1}{j+m+1}+\frac{(m+1)(m+2)}{(j+m+1)(j+m+2)}+\dots$$
$$=\frac{(j+m)!}{m!}\frac{1}{j-1}\frac{1}{(m+1)(m+2)\dots(m+j-1)}$$

where the latter equality is based on Theorem 131 of Knopp (1947, p. 233).

Rewriting, we have

$$\sum_{n=0}^{\infty} | a_{n,j+m} | = j + \frac{j^2}{j+m}\cdot\frac{j+m}{j-1} = j + \frac{j^2}{j-1},$$

$m = 1, 2, 3, \dots$. Let $M = \max\left(j + \frac{j^2}{j-1}, \sum_{n=0}^{\infty} | a_{n0} |, \dots, \sum_{n=0}^{\infty} | a_{nj} |\right)$. Then

$\sum_{n=0}^{\infty} | a_{nk} | \leqslant M$, independent of k, and by the condition given in Section 2 the sequence x is in l_1. Therefore $\frac{1}{j} I - C_1 \in III_1$ and this concludes the proof of the theorem.

Theorem 4 — If $\mathrm{Re}\,\frac{1}{\lambda} = 1, \lambda \neq 1$, then $\lambda \in II_2\sigma(C_1)$.

PROOF : Let λ be a fixed complex number such that $\mathrm{Re}\,\frac{1}{\lambda} = 1, \lambda \neq 1$. Since $\lambda I - C_1$ is a triangle it is one-to-one and $\lambda I - C_1 \in 1 \cup 2$. We look next at the conjugate operator $\lambda I - C_1'$. In Theorem 2 we saw that if $(\lambda I - C_1')\,x = 0$, then

$$x_0 = 0 \text{ and } x_n = \binom{\frac{1}{\lambda} - 1}{n - 1}(-1)^{n-1}x_1, \; n = 2, 3, 4, \dots .$$

Now

$$\sum_{n=0}^{\infty} |\, x_n \,| = |\, x_1 \,| \sum_{n=0}^{\infty} \left| \binom{\frac{1}{\lambda} - 1}{n} (-1)^n \right| = |\, x_1 \,| \sum_{n=0}^{\infty} \left| \binom{\frac{1}{\lambda} - 1}{n} \right| .$$

The latter series is divergent because $\operatorname{Re}\left(\frac{1}{\lambda} - 1\right) = 0$ (Knopp 1947, p. 426). Therefore, $x \in l_1$ implies $x = 0$ and $\lambda I - C_1' \in 1 \cup 2$.

If we now consult the state diagram, we see that $\lambda I - C_1 \in I_1 \cup II_2$. The case $\lambda I - C_1 \in I_1$ can be ruled out because $\sigma(C_1)$ is a closed set. This forces us to conclude that $\lambda I - C_1 \in II_2$.

Theorem 5 — $1 \in III_3\sigma(C_1)$.

PROOF: First observe $I - C_1$ is not one-to-one because $(I - C_1)\,1 = 0$. It is also trivial that $I - C_1 \in III$. For, if $z = (z_0, z_1, z_2, \dots)$ is an element of c such that $z_0 \neq 0$, then $\|(I - C_1)\,x - z\| \geqslant |\, z_0 \,| > \frac{|\, z_0 \,|}{2}$ for all $x \in c$. Hence z is not contained in $\overline{R(I - C_1)}$.

With the remark that $\operatorname{Re} \frac{1}{\lambda} = 1$ if and only if $\left|\lambda - \frac{1}{2}\right| = \frac{1}{2}$ we can summarize the results of this section by means of Fig. 2. The points in the interior of the

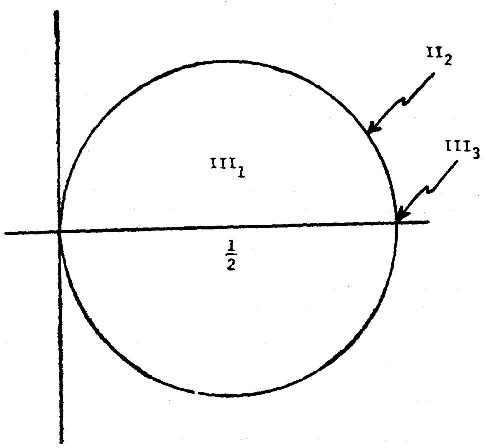

FIG. 2. The fine spectrum of C_1.

circle make up $III_1\sigma(C_1)$. The points on the circumference of the circle, except 1, make up $II_2\sigma(C_1)$. The set $III_3\sigma(C_1) = \{1\}$ and $\rho(C_1)$ consists of all points exterior to the circle.

4. The Fine Spectrum of C_1^p

It is a well-known fact that the spectrum of T^p, where T is a continuous linear operator on a Banach space and p is a positive integer, can be obtained from the spectrum of T by means of a mapping on the complex plane. Furthermore, a knowledge of the fine spectrum of T can be used to obtain a knowledge of the fine spectrum of T^p. Essentially, every "bad" property of $\lambda I - T$ is passed on to $\mu I - T^p$, where $\mu = \lambda^p$. In this section we use these ideas to obtain the fine spectral structure of C_1^p, $p = 2, 3, 4, \ldots$.

The following lemma is true for any operator on a Banach space but we state it in terms of C_1. The proof is elementary and is omitted.

Lemma 1 — Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ denote the p distinct pth roots of μ. Then $\mu I - C_1^p$ is one-to-one if and only if $\lambda_i I - C_1$ is one-to-one for each i. Further, $\mu I - C_1^p$ has a continuous inverse if and only if $\lambda_i I - C_1$ has a continuous inverse for each i.

In the next several theorems we derive the fine spectrum of C_1^p for a fixed integer $p \geqslant 2$.

Theorem 6 — Let $D = \left\{\lambda = \rho e^{i\phi} : \rho = \cos\phi \text{ and } -\frac{\pi}{p} < \phi < 0 \text{ or } 0 < \phi \leqslant \frac{\pi}{p}\right\}$ and let $D' = \{\mu = re^{i\theta} : \mu = \lambda^p \text{ and } \lambda \in D\}$. For each $\mu \in D'$, $\mu I - C_1^p \in II_2$.

Proof : As in Lemma 1 we denote the p distinct pth roots of μ by $\lambda_1, \lambda_2, \ldots, \lambda_p$. If θ satisfies the inequality $-\pi < \theta < 0$ or $0 < \theta < \pi$, so that ϕ satisfies either $-\frac{\pi}{p} < \phi < 0$ or $0 < \phi < \frac{\pi}{p}$, then one and only one of the λ's is in $II_2\sigma(C_1)$. The others belong to $\rho(C_1)$. Denote the λ in $II_2\sigma(C_1)$ by λ_1 and write

$$\mu I - C_1^p = (\lambda_1 I - C_1)(\lambda_2 I - C_1)\ldots(\lambda_p I - C_1)(-1)^{p+1}.$$

From Lemma 1 we have $\mu I - C_1^p \in 2$ and it is clear also, that $R(\mu I - C_1^p) = R(\lambda_1 I - C_1)$. Therefore, $\mu I - C_1^p \in II_2$.

When $\theta = \pi$ and $\phi = \frac{\pi}{p}$, there are two λ's in $II_2\sigma(C_1)$ with the other λ's in $\rho(C_1)$. Denote these λ's by λ_1 and λ_2 and write

$$-\rho^p I - C_1^p = (\lambda_1 I - C_1)(\lambda_2 I - C_1)\ldots(\lambda_p I - C_1)(-1)^{p+1}.$$

As was the case above, $\mu I - C_1^p \in 2$. Because each of $\lambda_1 I - C_1$ and $\lambda_2 I - C_1$ has a dense range it is also true that $(\lambda_1 I - C_1)(\lambda_2 I - C_1)$ has a dense range. Hence $-\rho^p I - C_1^p$ has a dense range so that $-\rho^p I - C_1^p \in II_2$.

Theorem 7 — Let $E = \left\{\lambda = \rho e^{i\phi} : \rho < \cos\theta \text{ and } -\frac{\pi}{p} < \phi \leqslant \frac{\pi}{p}\right\}$ and let $E' = \{\mu = re^{i\theta} : \mu = \lambda^p \text{ and } \lambda \in E\}$. Then for each $\mu \in E'$, $\mu I - C_1^p\ III_1 \cup III_2$; more precisely, $\mu I - C_1^p \in III_1$ if none of the pth roots of μ belongs to $II_2\sigma(C_1)$ and $\mu I - C_1^p \in III_2$ if at least one of the pth roots of μ belongs to $II_2\sigma(C_1)$.

PROOF: We write

$$\mu I - C_1^p = (\lambda_1 I - C_1)(\lambda_2 I - C_1)\ldots(\lambda_p I - C_1)(-1)^{p+1}$$

where factors containing λ's which belong to $III_1\sigma(C_1)$ are written first. Now λ_1 is in $III_1\sigma(C_1)$ because at least one λ belongs to $E \subset III_1\sigma(C_1)$. Since $R(\mu I - C_1^p) \subset R(\lambda_1 I - C_1)$ it follows that $\mu I - C_1^p \in III$.

That $\mu I - C_1^p \in 2$ or 1, according as μ has pth roots which belong or do not belong to $II_2\sigma(C_1)$, follows immediately from Lemma 1.

Theorem 8 — Let $F = \left\{\lambda = \rho e^{i\phi} : \rho > \cos\phi \text{ and } -\frac{\pi}{p} < \phi \leqslant \frac{\pi}{p}\right\}$ and let $F' = \{\mu = re^{i\theta} : \mu = \lambda^p \text{ and } \lambda \in F\}$, then for all μ in F', $\mu I - C_1^p \in I_1$.

PROOF: In this case all of the pth roots of μ belong to $\rho(C_1)$. Since $R(\lambda_i I - C_1) = c$ for each i, $R(\mu I - C_1^p) = c$. Using Lemma 1 we conclude $\mu I - C_1^p \in 1$.

Theorem 9 — $I - C_1^p \in III_3$.

PROOF : Observe that

$$(I - C_1^p) = (I - C_1)(\lambda_2 I - C_1)\ldots(\lambda_p I - C_1)(-1)^{p+1}$$

where $\lambda_2, \lambda_3, \ldots, \lambda_p$ all belong to $\rho(C_1)$. Now $I - C_1^p \in III$ because $R(I - C_1^p) = R(I - C_1)$ and $I - C_1^p \in 3$ by Lemma 1.

The results of this section can be summarized with reference to the closed curve given in polar coordinates by $r = \cos^p(\theta/p)$. This is the image of $\rho = \cos\phi$, $-\pi/p < \phi \leqslant \pi/p$, under the mapping $\lambda \to \lambda^p$.

The closed region bounded by the closed curve whose eqution is $r = \cos^p(\theta/p)$ is precisely $\sigma(C_1^p)$. If μ is on the boundary of this region, then $\mu \in II_2\sigma(C_1^p)$, except for 1, which belongs to and is all of $III_3\sigma(C_1^p)$. If μ is in the interior of this region, then $\mu \in III_1\sigma(C_1^p) \cup III_2\sigma(C_1^p)$. The results for C_1^2, C_1^3, and C_1^4 are illustrated in Figs. 3, 4, and 5, respectively.

5. AN APPLICATION TO SUMMABILITY

An important summability theorem, due to Mercer, states that if β is a complex number such that Re $\beta > 0$ and $x = \{x_n\}$ is a sequence of complex numbers, then $\lim (\beta x + (1 - \beta) C_1 x) = t$ implies $\lim x = t$. Theorems of this type, no matter what summability operator is used, have come to be called Mercerian theorems (Hardy 1949).

In this section we obtain Mercerian theorems for all of the summability operators studied in this paper. Many of these results are known but it is our purpose to show how they can be obtained quickly from a knowledge of fine spectral structure of the operators.

We set the stage with two lemmas. The proof of Lemma 2 is elementary. Lemma 3 is due to Wilansky (1964).

Lemma 2 — Suppose T is a regular matrix and x is any sequence of complex numbers. Then for $\beta = 1$ and for complex β such that $\beta/(\beta - 1) \in \rho(T)$, $\lim (\beta x + (1 - \beta)Tx) = t$ implies $\lim x = t$.

Lemma 3 — Let T be a conservative matrix which is one-to-one on c. Then T sums no bounded divergent sequence if and only if $R(T)$ is closed in c.

For proof of Lemma 3 see Wilansky (1964, p. 247).

Our next theorem includes the classical theorem by Mercer and More.

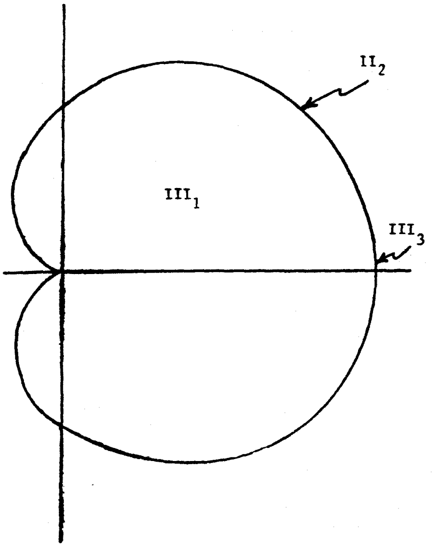

FIG. 3. The fine spectrum of C_1^2.

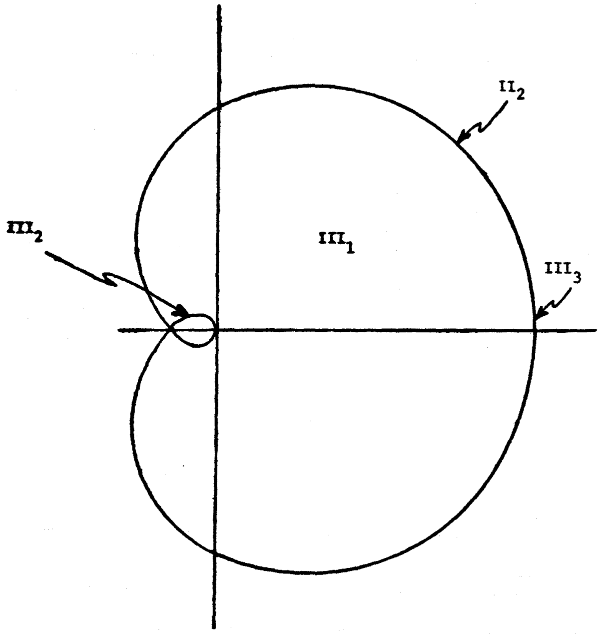

FIG. 4. The fine spectrum of C_1^3.

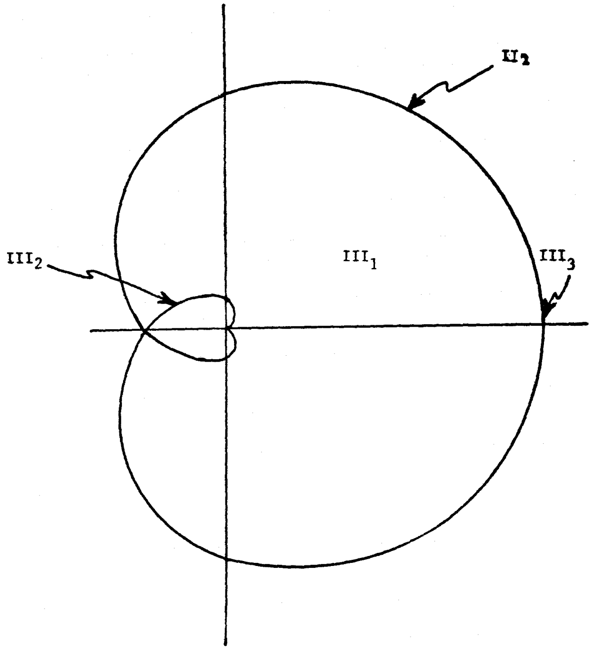

FIG. 5. The fine spectrum of C_1^4.

Theorem 10 — (a) If Re $\beta > 0$ and lim $(\beta x + (1-\beta) C_1 x) = t$, then lim $x = t$;

(b) If Re $\beta < 0$ and lim $(\beta x + (1-\beta) C_1 x) = t$, then lim $x = t$ or x is unbounded;

(c) If Re $\beta = 0$, then the operator $\beta I + (1-\beta) C_1$ sums bounded divergent sequences.

PROOF : By Lemma 2 the conclusion to (a) holds for $\beta = 1$ and for β such that $\frac{\beta}{\beta - 1}$ belongs to $\rho(C_1)$. But $\frac{\beta}{\beta-1}$ belongs to $\rho(C_1)$ if and only if Re $\frac{\beta-1}{\beta} < 1$ (see Theorem 1). The latter inequality is equivalent to Re $\beta > 0$.

Next observe that Re $\beta < 0$ if and only if Re $\frac{\beta-1}{\beta} > 1$. Let β be a complex number which satisfies these inequalities. By Theorems 2 and 3 it follows that $\frac{\beta}{\beta-1} \in III_1\sigma(C_1)$. If $\beta x + (1-\beta) C_1 x \in R(\beta I + (1-\beta) C_1)$, then $x \in c$ and since $\beta I + (1-\beta) C_1$ is regular we can conclude lim $x = t$. If $\beta x + (1-\beta) C_1 x \in c - R(\beta I + (1-\beta) C_1)$, then x is unbounded because $R(\beta I + (1-\beta) C_1)$ is closed and by Lemma 3, x cannot be a bounded divergent sequence. This is the conclusion in (b).

If $\beta \neq 0$, then $\operatorname{Re} \beta = 0$ if and only if $\operatorname{Re} \frac{\beta - 1}{\beta} = 1$. Employing Theorem 4 we conclude that $\beta I + (1 - \beta) C_1 \in II_2$. This means, among other things, that $[\beta I + (1 - \beta) C_1]^{-1}$ is discontinuous. Therefore $R(\beta I + (1 - \beta) C_1)$ is not closed and by Lemma 3, $\beta I + (1 - \beta) C_1$ sums bounded divergent sequences. When $\beta = 0$, $\beta I + (1 - \beta) C_1$ reduces to C_1 and it is well-known that this operator sums bounded divergent sequences. The proof is now complete.

The last theorem provides Mercerian type results for C_1^p.

Theorem 11 — Let p be a fixed positive integer.

(a) If $\left|\frac{\beta}{\beta - 1}\right| > \cos^p \frac{\arg \frac{\beta}{\beta - 1}}{p}$ and $\lim (\beta x + (1 - \beta) C_1^p x) = t$, then $\lim x = t$;

(b) If $\left|\frac{\beta}{\beta - 1}\right| < \cos^p \frac{\arg \frac{\beta}{\beta - 1}}{p}$, $\frac{\beta}{\beta - 1} \neq z^p$ where $z = \rho e^{i\phi}$ is a complex number such that $\rho = \cos \phi$, and $\lim (\beta I + (1 - \beta) C_1^p x) = t$, then $\lim x = t$ or x is unbounded;

(c) For all other values of β, that is, those values for which $\frac{\beta}{\beta - 1} = z^p$, where $z = \rho e^{i\phi}$ is a complex number such that $\rho = \cos \phi$, $\beta I + (1 - \beta) C_1^p$ sums bounded divergent sequences.

PROOF: The conclusions follow readily, using the same arguments as those in Theorem 10 and the fine spectral structure of C_1^p as given in Section 4.

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