A NOTE ON THE STUFE OF QUADRATIC FIELDS

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The stufe of all imaginary quadratic fields is determined by very elementary considerations.

Let F be a field. The stufe of F is defined as the least positive integer $s(F)$, if it exists, such that $-1 = a_1^2 + \ldots + a_s^2$, $a_i \in F$. If no such s exists we say that F is formally real and that the stufe of F is infinite. A well-known theorem of Pfisters says that the stufe of any field, if it exists finite, is a power of 2. In a recent issue of the Journal of Number Theory (vol. 3, number 3, August 1971, pp. 310-15), the stufe of all imaginary quadratic fields is determined as a corollary to a general theorem. There is a very simple way of proving this directly (our enunciation of it is more lucid).

Theorem — Let $D > 0$ be a square-free integer. Then the stufe $s(k)$ of $k = Q(\sqrt{-D})$ is

$$\begin{cases} 1, & \text{if } D = 1, \\ 2, & \text{if } D \neq 1 \text{ and } D \text{ is not of the form } 8b+7 \text{ (i.e. if } D \text{ is expressible as a sum of} \leqslant 3 \text{ squares),} \\ 4, & \text{if } D \text{ is of the form } 8b+7 \text{ (i.e. if } D \text{ cannot be expressed as a sum of less than 4 squares).} \end{cases}$$

PROOF : Writing $D = a^2 + b^2 + c^2 + d^2$, $(a, b, c, d \in Z)$ we see that

$$0 = (\sqrt{-D})^2 + a^2 + b^2 + c^2 + d^2,$$

and hence for any $k = Q(\sqrt{-D})$, $s(k) \leqslant 4$. Notice next that $s(k) = 1$ if and only if $i = \sqrt{-1} \in k$; this happens only in the case $D = 1$.

If $D \not\equiv 7 \pmod 8$ then D is a sum of 3 squares and so

$$0 = (\sqrt{-D})^2 + D = (\sqrt{-D})^2 + 3 \text{ squares} = \text{a sum of 4 squares.}$$

Hence $s(k) \leqslant 3$ and therefore $s(k) = 1$ or 2 for

$$-1 = a^2 + b^2 + c^2 \Rightarrow -1 = \left(\frac{xz + y}{x^2 + y^2}\right)^2 + \left(\frac{yz - x}{x^2 + y^2}\right)^2.$$

If in addition, $D \neq 1$, then $s(k) = 2$.

Finally let $D \equiv 7 \pmod 8$. If $s(k)$ were < 4 then it would be equal to 2 and so

$$-1 = (a_1 + b_1\sqrt{-D})^2 + (a_2 + b_2\sqrt{-D})^2,\ a_1, b_1, a_2, b_2, \in Q.$$

Here not both b_1, b_2 are 0; without loss of generality let $b_1 \neq 0$. Equating reals and imaginaries gives

$$a_1^2 + a_2^2 - D(b_1^2 + b_2^2) = -1$$
$$a_1b_1 + a_2b_2 = 0.$$

These imply $D = (a_2/b_1)^2 + (b_1/b_1^2 + b_2^2)^2 + (b_2/b_1^2 + b_2^2)^2$. Thus D is a sum of 3 rational squares, a contradiction since $D \equiv 7 \pmod 8$. Hence $s(k) \nless 4$ and therefore $s(k) = 4$. This completes the proof of the theorem.