SIMILARITY ANALYSIS OF FINGERO-IMBIBITION IN DOUBLE PHASE FLOW THROUGH UNDERGROUND POROUS MEDIUM

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The present paper analytically discusses the phenomenon of fingero-imbibition in double phase flow through porous media by using similarity technique. Fingero-imbibition is a physical phenomenon which denotes the simultaneous occurrence of two special phenomenon viz., Fingering and Imbibition in a porous media. A possible analytical form for the saturation distribution in the investigated problem, which represents the average cross-sectional area occupied by the fingers have been suggested. For the particular case, when the saturation coefficient is regarded as a constant the displacing phase saturation distribution in terms of hypergeometric functions have been obtained.

1. INTRODUCTION

Fingero-Imbibition is the simultaneous occurrence of two special phenomena viz., fingering and imbibition. It is known that if a porous medium filled with some fluid is brought into contact with another fluid, which preferentially wets the medium, then there is a spontaneous flow of the wetting fluid into the medium and a counter-flow of the resident fluid from the medium. This phenomenon is called imbibition (Scheidegger 1960a). Again, when a fluid contained in a porous medium is displaced by another of lesser viscosity, instead of a regular displacement of the whole front, protuberances may occur that shoot through the porous medium at relatively great speeds. These protuberances are called fingers (Scheidegger 1961). The phenomena of fingering and imbibition, and the flow of two immiscible fluids through porous media have gained considerable current interest due to their frequent occurrence in problems of petroleum technology. Many authors have investigated these problems from different view-points; for example, fingering by Scheidegger (1969, 1960) and Verma (1968, 1969a, 1969b, 1969c), imbibition by Graham and Richardson (1959), Rijik (1960) and Verma (1969d). Recently Verma (1970a), (1970b), has obtained a perturbation solution of fingero-imbibition phenomena. In this paper we discuss some aspects of the problem of fingero-imbibition by using a similarity technique.

The basic assumptions underlying the present investigation are clearly stated in the section on statement of the problem. The nonlinear differential equation of fingeroimbibition has been solved by similarity analysis of one parameter group transformations. A possible analytical solution is obtained in investigated problem and a mathematical form of saturation distribution is obtained in the particular case when the saturation coefficient is constant.

2. Theoretical Formulation of the Problem

We consider here that a semi-infinite cylindrical piece of homogeneous porous matrix containing native liquid N, is completely surrounded by an impermeable surface except for one end of the cylinder which is labelled as the imbibition face and this end is exposed to an adjacent formation of injected liquid I. It is assumed that the liquid I is the preferentially wetting and lesser viscous phase. This arrangement gives rise to a displacement process in which the injection of liquid I is initiated by imbibition and the consequent displacement of native liquid N produces fingering. We consider that this arrangement furnishes the investigated problem of fingeroimbibition.

For the mathematical formulation of the above model we assume that the flow is governed by Darcy's law and consider only the statistical behaviour of fingers. Thus the seepage velocities of liquid I and N are expressed as

$$V_i = -\frac{K_i}{\gamma_i} K \frac{\partial p_i}{\partial x} \qquad \ldots(2.1)$$

$$V_n = -\frac{K_n}{\gamma_n} K \frac{\partial p_n}{\partial x} . \qquad \ldots(2.2)$$

The symbols are defined in the nomenclature. It may be mentioned that the statistical treatment of fingers (Verma 1969a) is formally identical to the Buckley-Leverett description of two immiscible fluids flow and the displacing phase saturation is defined by the average cross-sectional area occupied by fingers.

The condition of linear counter-current imbibition, viz., $V_i = - V_n$ yields

$$\frac{K_i}{\gamma_i} \frac{\partial p_i}{\partial x} + \frac{K_n}{\gamma_n} \frac{\partial p_n}{\partial x} = 0. \qquad \ldots(2.3)$$

The capillary pressure (p_c) is defined as the pressure discontinuity between the flowing phases and may be written as

$$p_c = p_n - p_i, \qquad \ldots(2.4)$$

Eliminating p_n between eqns. (2.3) and (2.4), we get

$$\left[\frac{K_i}{\gamma_i} + \frac{K_n}{\gamma_n}\right] \frac{\partial p_i}{\partial x} + \frac{K_n}{\gamma_n} \frac{\partial p_c}{\partial x} = 0 . \qquad \ldots(2.5)$$

Combining eqns. (2.1) and (2.5), we have

$$V_i = K \frac{\frac{K_i}{\gamma_i} \frac{K_n}{\gamma_n}}{\frac{K_i}{\gamma_i} + \frac{K_n}{\gamma_n}} \frac{\partial p_c}{\partial x}. \quad \ldots(2.6)$$

We may write the equation of continuity for injected water as

$$P \frac{\partial S_i}{\partial t} + \frac{\partial V_i}{\partial x} = 0. \quad \ldots(2.7)$$

Substituting the value of V_i from eqn. (2.6) into (2.7), we obtain,

$$P \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x} \left[K \frac{K_i K_n}{\gamma_n K_i + \gamma_i K_n} \frac{dp_c}{dS_i} \frac{\partial S_i}{\partial x} \right] = 0. \quad \ldots(2.8)$$

A set of appropriate boundary conditions may be chosen as

$$S_i(0, t) = S_{i_0}, \; S_i(\infty, t) = 0 \quad \ldots(2.9)$$

where the first condition defines that S_{i_0} is the imbibition face saturation and the second condition states the fact that the saturation at infinity remains zero. Equation (2.8) together with conditions (2.9) constitute the desired differential system.

3. Mathematical Solution

Since all the parameters are function of the saturation, therefore, we can write eqn. (2.8) into the form

$$P \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x} \left((KD(S_i) \cdot \frac{\partial S_i}{\partial x} \right) = 0 \quad \ldots(3.1)$$

where

$$D(S_i) = \frac{K_i K_n}{\gamma_n K_i + \gamma_i K_n} \frac{dp_c}{dS_i}. \quad \ldots(3.2)$$

Equation (3.1) can be rewritten in the dimensionless form by considering

$$\xi = \frac{X}{L}, \text{ and } \theta = \frac{K}{PL^2} t. \quad \ldots(3.4)$$

Substituting this value in eqn. (3.1), we get

$$\frac{\partial S_i}{\partial \theta} + \frac{\partial}{\partial \xi} \left(D(S_i) \frac{\partial S_i}{\partial \xi} \right) = 0, \quad \ldots(3.5)$$

This is the nonlinear differential equation of the saturation together with boundary condition

$$S_i(0, \theta) = S_{i_0}, \; S_i(\infty, \theta) = 0, \quad \ldots(3.6)$$

For obtaining the exact solution of eqn. (3.5) subject to boundary condition (3.6) we choose a similarity variable

$$\phi = \frac{\xi}{\sqrt{\theta}} \text{ (Boltzman 1894).} \qquad \ldots(3.7)$$

When this transformation is applied to eqn. (3.5), we obtain an ordinary differential equation of the form

$$\frac{d}{d\phi}\left[D(S_i) \frac{dS_i}{d\phi} \right] - \tfrac{1}{2}\phi \frac{dS_i}{d\phi} = 0, \qquad \ldots(3.8)$$

The variation of S_i between zero and one suggests that after first integration S_i may be used as the independent variable. By integrating eqn. (3.8) between the limit 0 to S_i, we get

$$D(S_i) = \tfrac{1}{2} \frac{d\phi}{dS_i} \int_0^{S_i} \phi \, dS_i. \qquad \ldots(3.9)$$

Since the pressure discontinuity between two flowing phases (capillary pressure) may be regarded as purely function of the displacing saturation (Scheidegger 1960b, pp. 54, 55), therefore, it can be written

$$S_i = \delta(p_c). \qquad \ldots(3.10)$$

Then equation (3.9) can be written in terms of capillary pressure as

$$D'(p_c) = \tfrac{1}{2} \frac{d\phi}{dp_c} \int_{\infty}^{\delta(p_c)} \phi'(p_c) \, dp_c \qquad \ldots(3.11)$$

where

$$\phi'(p_c) = \phi\delta'(p_c)$$

The further solution of eqn. (3.11) can be obtained by applying the Laplace transformation. By assuming that in particular case $\frac{d\phi}{dp_c}$ and $\int_{\infty}^{\delta(p_c)} \phi'(p_c)\, dp_c$ are the convolution of each other then by the well-known convolution theorem, we get

$$D'(p_c) = \tfrac{1}{2} L\left\{ \frac{d\phi}{dp_c} \int_{\infty}^{\delta(p_c)} \phi'(p_c) \, dp_c \right\}$$

$$= \tfrac{1}{2} L\left\{ \frac{d\phi}{dp_c} \right\} L\left\{ \int_{\infty}^{\delta(p_c)} \phi'(p_c) \, dp_c \right\}$$

$$\bar{D}'(s) = \tfrac{1}{2}\{s\bar{\phi} - \phi(0)\}\bar{\phi}'$$

$$\phi(0) = 0.$$

Therefore,

$$s\bar{\phi}\ \bar{\phi}' = 2\,\bar{D}'(s)$$

$$\bar{\phi}\ \bar{\phi}' = \frac{2}{s}\ \bar{D}'(s).$$

Then by inverse Laplace transformation

$$\phi\,\phi' = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{s}\,p_c\left(\frac{2}{s}\ \bar{D}'(p_c)\,dp_c.\right. \qquad \ldots(3.12)$$

Equation (3.12) gives the capillary pressure function between two flowing phases.

4. Particular Case : Solution for Constant $D(S_i)$

We assume here that the average value of $D(S_i) \equiv \bar{D}(S_i)$ is a constant. In this case the similarity solution can be obtained as below. The equation (3.5) can be written as

$$\frac{\partial S_i}{\partial \theta} + \bar{D}\ \frac{\partial^2 S_i}{\partial \xi^2} = 0. \qquad \ldots(4.1)$$

To solve eqn. (4.1), we use Birkhoff's technique of one parameter group transformations (Scheidegger 1960). Let a group T_1 consisting of a set of transformations be defined as

$$T_1 : \bar{\xi} = a^{p}\xi,\ \bar{\theta} = a^{r}\theta,\ \text{and}\ \bar{S}_i = a^{s}S_i, \qquad \ldots(4.2)$$

where the parameter $a \neq 0$ and p, r, s are real numbers to be determined.

Substituting the values from eqn. (4.2) in equation (4.1), we obtain

$$a^{r-s}\ \frac{\partial \bar{S}_i}{\partial \bar{\theta}} + \bar{D}\ a^{2p-s}\ \frac{\partial^2 \bar{S}_i}{\partial \bar{\xi}^2} = 0. \qquad \ldots(4.3)$$

Equation (4.3) is absolute conformal invariant under T_1 provided

$$2p - s = r - s\ . \qquad \ldots(4.4)$$

Now, we choose to eliminate 'θ' so that the solution of eqn. (4.4) for $r \neq 0$ is equivalent to the solution of

$$2\,\frac{p}{r} - \frac{s}{r} = 1 - \frac{s}{r}\,. \qquad \text{...(4.5)}$$

Choosing an arbitrary constant A and then setting

$$\frac{s}{r} = A \qquad \text{...(4.6)}$$

combining eqn. (4.6) with (4.5), we get

$$\frac{p}{r} = \tfrac{1}{2}. \qquad \text{...(4.7)}$$

Thus the invariants of the group T_1 are given by

$$\eta = \frac{\xi}{\sqrt{\theta}}\,,\ F(\eta) = \frac{S_i(\xi, \theta)}{\theta^A}\,, \qquad \text{...(4.8)}$$

and the derivatives of the saturation S_i in terms of $F(\eta)$ are

$$\left.\begin{aligned} \frac{\partial S_i}{\partial \theta} &= \theta^{A-1}\{AF - \tfrac{1}{2}\,\eta F'\} \\ &\text{and} \\ \frac{\partial^2 S_i}{\partial \xi^2} &= \theta^{A-1}\,F''. \end{aligned}\right\} \qquad \text{...(4.9)}$$

Substituting these values in eqn. (4.1) we get

$$\theta^{A-1}\,[(AF - \tfrac{1}{2}\,\eta F') + \bar{D}F''] = 0. \qquad \text{...(4.10)}$$

Since $\theta^{A-1} \neq 0$, therefore, we have

$$F'' + \alpha\eta F' + bF = 0, \qquad \text{...(4.11)}$$

where

$$\alpha = -\,\frac{1}{2\bar{D}}$$

and

$$b = A/\bar{D}\,.$$

This is linear ordinary differential equation of second order whose solution can be given by the following substitution.

$$F(\eta) = u(Z);\ 2Z = -\,\alpha\eta^2. \qquad (4.12)$$

By substituting equation (4.12) in equation (4.11), we get

$$Zu''(Z) + (\tfrac{1}{2} - Z)\,u'(Z) - \frac{b}{2\alpha}\,u(Z) = 0, \qquad \text{...(4.13)}$$

This is called confluent hypergeometric differential equation (Murphy 1969), whose general solution is given by

$$u(Z) = C_1 u_1 + C_2 u_2 \qquad \text{...(4.14)}$$

where C_1 and C_2 are constants and

$$u_1 = {}_1F_1(-A, \tfrac{1}{2}; Z);$$

and

$$u_2 = \sqrt{Z}\, {}_1F_1(\tfrac{1}{2} - A, \tfrac{3}{2}; Z).$$

The values of ${}_1F_1(-A, \frac{1}{2}; Z)$ and ${}_1F_1(\frac{1}{2} - A, \frac{3}{2}; Z)$ are respectively

$${}_1F_1(-A; \tfrac{1}{2}; Z) = \sum_{k=0}^{\infty} B_k Z^k$$

$$B_k = \frac{-A(1-A)\ldots(k-A)}{\frac{1}{2}\cdot\frac{3}{2}\ldots(k+\frac{1}{2})\,k!}$$

and

$${}_1F_1(\tfrac{1}{2} - A, \tfrac{3}{2}; Z) = \sum_{k=0}^{\infty} C_k Z^k,$$

$$C_k = \frac{(\frac{1}{2}-A)(\frac{3}{2}-A)\ldots(k-A+\frac{1}{2})}{\frac{3}{2}\cdot\frac{5}{2}\ldots(\frac{3}{2}+k)k!}$$

where A is free parameter and ${}_1F_1(\beta, \delta; Z)$ denotes the confluent hypergeometric function of argument Z and parameter β and δ.

Then the solution of equation (4.13) in terms of hypergeometric function is

$$u(Z) = C_1\, {}_1F_1(-A, \tfrac{1}{2}; Z) + C_2\sqrt{Z}\, {}_1F_1(\tfrac{1}{2} - A, \tfrac{3}{2}; Z). \qquad \text{...(4.15)}$$

Equation (4.15) gives a formal analytical solution of the nonlinear differential equation for fingero-imbibition under the specifications of the investigated problem. The constant C_1 and C_2 may be evaluated from particular specifications for the problem.

5. Conclusion

In this paper we have obtained the analytical solution of the nonlinear differential equation of Fingero-Imbibition phenomenon by using a similarity technique and the possibility for deriving an expression for the wetting phase saturation in exact form has been discussed. In particular, we have obtained the saturation of the wetting phase when the saturation coefficient is assumed to be constant, which represents the average cross-sectional area occupied by the fingers. In this case the author has not included the numerical and graphical illustrations due to his particular interest in analytical discussion.

Notwithstanding the limitations of the present analysis it is believed that the similarity solution for a complicated flow problem in porous media which is obtained here will have relevance to some physical problems useful in further analytical study.

NOMENCLATURE

A = free parameter

C = capillary pressure constant

K = permeability of the system

K_i = fictitious relative permeability of injected water

K_n = fictitious relative permeability of native water

P = porosity of the medium

p_i = pressure of injected water

p_n = pressure of native water

S_i = saturation of injected water

S_n = saturation of native water

t = time

V_i = seepage velocity of injected water

V_n = seepage velocity of native water

x = linear coordinate

β = capillary pressure coefficient

γ_i = viscosity of injected water

γ_n = viscosity of native water

m = viscosity ratio.

REFERENCES

Birkhoff, G. (1960). Hydrodynamics. Princeton University Press, Princeton, New Jersey, (Chapter 4 and 5).

Graham, J. W., and Richardson, J. G. (1959). Theory and application of imbibition phenomena in recovery of oil. *Trans. A.I.M.E.*, **216**, 377.

Murphy, G. M. (1969). Ordinary Differential Equations and Their Solutions. East-West Press, New Delhi, pp. 323, 331.

Rijik, V. M. (1960). O Kapilliarnoi proptke Vodoineft Cenasisennogo ghidrofilnogo plasta. *Izv. Akad. Nauk SSSR, Otd. Tekhn. Nauk, Mekhan. i Mashinostr*, 2.

Scheidegger, A. E. (1960a). The Physics of Flow through Porous Media. University of Toronto Press, pp. 201, 216, 229.

———(1960b). General spectral theory for the onset of Stabilities in displacement processes n porous media. *Geofis. Pura Appl.*, **47**, 1, 41.

Scheidegger, A. E. (1961). Dynamic similarity of instabilities of displacement fronts in porous media. *Can. J. Phys.*, **39**, 2, 1251.

————(1969). Stability conditions for displacement processes in porous media. *Can. J. Phys.*, **47**, 2, 209.

Scheidegger, A. E., and Johnson, E. F. (1961). The statistical behaviour of instabilities in displacement process in porous media. *Can. J. Phys.*, **39**, 2, 326.

Verma, A. P. (1968). Motion of immiscible liquids in a cracked heterogeneous porous medium with capillary pressure. *Rev. Roumaine Sci. Tech. Mech. Appl.*, **13**, 2, 277-92.

————(1969a). Statistical behaviour of fingering in a displacement process in heterogeneous-porous medium with capillary pressure. *Can. J. Phys.*, **47**, 3, 319-24.

————(1969b). Stabilization of fingers in a slightly cracked heterogeneous porous medium. Fundamentals of transport phenomena in porous media. Elseviers, Amsterdam, 1972, pp. 221.

————(1969c). Statistical behaviour of fingering in a displacement process in heterogeneous porous medium with pressure dependent phase densities. *Rev. Roum. Sci. Tech., Mech. Appl.*, **14**, 3, 471-82.

————(1969d). Imbibition in a cracked porous medium *Can. J. Phys.*, **47**, 22, 2519-24.

————(1970a). Fingero Imbibition in Artificial Replenishment of Ground Water through Cracked Porous Medium. *Water Resources, Research*, 6, 3.

————(1970b). Fingero-imbibition in slightly dipping porous medium. *J. appl. Phys.*, **41**, 9, 3638.