ON A NONLINEAR INTEGRO-DIFFERENCE-DIFFERENTIAL EQUATION

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A boundedness theorem concerning the solutions of a nonlinear integro-difference-differential equation is proved.

§1. In this paper, we consider the nonlinear integro-difference-differential equation of the type

$$z'(t) = A(t)\,z(t) + B(t)\,z(t-1) + f\left(t, z(t), z(t-1), \int_0^t g\left(s, z(s), z(s-1)\right)ds\right) \qquad ...(1.1)$$

for $0 \leqslant t < \infty$ under the initial conditions

$$z(t-1) = \phi(t)\ (0 \leqslant t < 1) \text{ and } z(0) = z_0. \qquad ...(1.2)$$

Here $A(t)$ and $B(t)$ are continuous for $0 \leqslant t < \infty$; $f(t, z, u, v)$ is continuous for $0 \leqslant t < \infty$, $|z| < \infty$, $|u| < \infty$, $|v| < \infty$, $g(t, z, u)$ is continuous for $0 \leqslant t < \infty$, $|z| < \infty$, $|u| < \infty$; $\phi(t)$ is continuous for $0 \leqslant t < 1$, and $\lim_{t \to 1-0} \phi(t) = \phi(1-0)$ exists. The kernel function for the equation

$$z'(t) = A(t)\,z(t) + B(t)\,z(t-1) \qquad ...(1.3)$$

will be denoted by $K(t, s)$. For the properties of the kernel function and the integral representation of the solution of (1.1) when the integral term in f is absent, see, Bellman and Cooke (1963). It is supposed that the existence and uniqueness of the solution of (1.1) with (1.2) are guaranteed for $0 \leqslant t < \infty$.

Considerable literature on the study of integro-differential equations has been published during the past few years, see for example, Grossman and Miller (1970), Levin (1969), Londen (1969), Nohel (1967) and some of the references given there. Although the boundedness theorems on the integro-differential equations of various types are available, eqn. (1.1) considered in this paper appears to be new.

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§2. Concerning eqn. (1.1) we prove the following theorem.

Theorem — In eqn. (1.1) we suppose that the following conditions are satisfied:

(i) the unique solution $z_0(t)$ of (1.3) with (1.2) is bounded;

(ii) $| f(t, z, u, v) | \leqslant h(t) (| z | + | u | + | v |)$,

where $h(t)$ is continuous for $0 \leqslant t < \infty$ and

$$\int_0^\infty h(t)\, dt < \infty; \qquad \text{...(2.1)}$$

(iii) $| g(t, z, u) | \leqslant l(t)(| z | + | u |)$,

where $l(t)$ is continuous for $0 \leqslant t < \infty$ and

$$\int_0^\infty l(t)\, dt < \infty; \qquad \text{...(2.2)}$$

(iv) the kernel function $K(t, s)$ is bounded, that is

$$| K(t, s) | \leqslant c \qquad (0 \leqslant s \leqslant t < \infty), \quad \text{where } c \text{ is constant.} \qquad \text{...(2.3)}$$

Then, the solution of (1.1) with (1.2) is bounded for $0 \leqslant t < \infty$.

The proof of our theorem relies on the following lemma which is a general version of the 'Gronwall-Bellman inequality' recently proved by the author (Pachpatte 1973).

Lemma — Let $u(t)$, $f(t)$ and $g(t)$ be real valued non-negative continuous functions defined on $I = [0, \infty)$, for which the inequality

$$u(t) \leqslant u_0 + \int_0^t f(s)\, u(s)\, ds + \int_0^t f(s) \left(\int_0^s g(\tau)\, u(\tau)\, d\tau\right) ds, \; t \in I$$

holds, where u_0 is a non-negative constant. Then

$$u(t) \leqslant u_0 \left[1 + \int_0^t f(s) \; \exp \left(\int_0^s (f(\tau) + g(\tau))\, d\tau\right) ds\right], \; t \in I.$$

Proof of the Theorem

Any solution of (1.1) with (1.2) is represented by

$$z(t) = z_0(t) + \int_0^t K(t, s)\, f(s, z(s), z(s-1), \int_0^s g(\tau, z(\tau), z(\tau - 1))\, d\tau)\, ds.$$

Now we have to consider two cases :

(I) *Case* $0 \leqslant t \leqslant 1$: It follows from (1.2), (ii), (iii) and (2.3) that

$$| z(t) | \leqslant | z_0(t) | + \int_0^t | K(t, s) | \; | f(s, z(s), z(s-1), \int_0^s g(\tau, z(\tau), z(\tau - 1)) \, d\tau) \, ds |$$

$$\leqslant c_1 + \int_0^t c\, h(s) \, [\, | z(s) | + | \phi(s) | \,] \, ds$$

$$+ \int_0^t c\, h(s) [\int_0^s l(\tau) \{ | z(\tau) | + | \phi(\tau) | \} \, d\tau] \, ds$$

$$\leqslant c_2 + \int_0^t c\, h(s) \, | z(s) | \, ds + \int_0^t c\, h(s) \, [\int_0^s l(\tau) \, | z(\tau) | \, d\tau] \, ds$$

where c_1 is the upper bound for $| z_0(t) |$ and

$$c_2 = c_1 + \int_0^1 c\, h(s) \, | \phi(s) | \, ds + \int_0^1 c\, h(s) \, [\int_0^s l(\tau) \, | \phi(\tau) | \, d\tau] \, ds.$$

Now applying lemma, this inequality leads us to

$$| z(t) | \leqslant c_2 \, [1 + \int_0^t c\, h(s) \, . \exp(\int_0^s \{c\, h(\tau) + l(\tau)\} \, d\tau) \, ds]$$

which in view of assumptions (2.1) and (2.2) implies that $| z(t) |$ is bounded.

(II) *Case* $1 \leqslant t < \infty$: It follows by (1.2), (ii), (iii) and (2.3) that

$$| z(t) | \leqslant | z_0(t) | + \int_0^1 | K(t, s) | \; | f(s, z(s), z(s-1), \int_0^s g(\tau, z(\tau), z(\tau - 1)) \, d\tau) | \, ds$$

$$+ \int_1^t | K(t, s) | \; | f(s, z(s), z(s-1), \int_0^s g(\tau, z(\tau), z(\tau - 1)) \, d\tau) | \, ds$$

$$\leqslant c_2 + \int_0^t c \, [h(s) + h(s+1)] \, | z(s) | \, ds$$

$$+ \int_0^t c\, h(s) \, (\int_0^s (l(\tau) + l(\tau + 1)) \, | z(\tau) | \, d\tau) \, ds$$

$$+ \int_1^t c\, h(s) \, [\int_0^1 l(\tau) \, | \phi(\tau) | \, d\tau] \, ds$$

$$\leqslant c_3 + \int_0^t c[h(s) + h(s+1)] \, | z(s) | \, ds$$

$$+ \int_0 c[h(s) + h(s+1)] \, (\int_0^s (l(\tau) + l(\tau + 1)) \, | z(\tau) | \, d\tau) \, ds$$

where

$$c_3 = c_2 + \int_1^t c\, h(s) \left[\int_0^1 l(\tau) \mid \phi(\tau) \mid d\tau \right] ds.$$

Now applying lemma, this inequality leads us to

$$\mid z(t) \mid \leqslant c_3 \Big[1 + \int_0^t c(h(s) + h(s+1))$$

$$\times \exp \Big(\int_0^s \{c(h(\tau) + h(\tau + 1)) + (l(\tau) + l(\tau + 1))\} \, d\tau \Big) \, ds\Big]$$

which in view of assumptions (2.1) and (2.2) implies that $\mid z(t) \mid$ is bounded. This completes the proof.

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