ON SOME INTEGRAL INEQUALITIES OF THE GRONWALL-BELLMAN TYPE

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In the present paper some integral inequalities of the Gronwall-Bellman type are proved which may be used as a tool in applications.

§1. On the basis of various motivations Gronwall-Bellman inequality has been extended and used considerably in various contexts. For instance, in obtaining bounds on the norm of solutions, in establishing uniqueness of solutions and in deriving conditions for stability of solutions of differential equations, such inequalities are handy tools. The recent work of Butler and Rogers (1971), Gollwitzer (1969), Lakshmikantham (1973), and Willett (1964) show that the interest in the Gronwall-Bellman inequality still continues. In a recent paper, the author (Pachpatte 1974) has obtained a useful general version of the Gronwall-Bellman inequality in the following form.

Theorem 1 — Let $u(t)$, $f(t)$, $g(t)$, $h(t)$ and $k(t)$ be real valued non-negative continuous functions defined on $I = [0, \infty)$ for which the inequality

$$u(t) \leqslant f(t) + g(t) \left[\int_0^t h(s)\, u(s)\, ds + \int_0^t h(s)\, g(s) \left(\int_0^s k(\tau)\, u(\tau)\, d\tau \right) ds \right]$$

holds for all $t \in I$. Then

$$u(t) \leqslant f(t) + g(t) \left[\int_0^t h(s) \{ f(s) + g(s) \exp \left(\int_0^s g(\tau) (h(\tau) + k(\tau))\, d\tau \right] \right.$$

$$\left. \times \int_0^s f(\tau) (h(\tau) + k(\tau)) \exp \left(- \int_0^\tau g(\eta) (h(\eta) + k(\eta))\, d\eta \right) d\tau \}\, ds \right]$$

for all $t \in I$.

Some of the applications of this theorem and its variants, to the stability and boundedness of solutions of Volterra integral equations and integro-differential equations of the more general type are given by Pachpatte (1974). In this paper, we

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wish to establish some useful integral inequalities which claim their origin to Theorem 1. Although the integral inequalities of the Gronwall-Bellman type are widely known and used, there appear to be no results of this kind in the literature.

§2. The most commonly used generalization of the Gronwall-Bellman inequality is due to Bihari (1956). Recent special uses of integral inequalities in which the associated integral equation is nonlinear in the integral have been considered in papers by Gollwitzer (1969) and Willett (1964). In Theorem 2 below we establish a general version of the inequality obtained by Gollwitzer in (1969).

Theorem 2 — Let $u(t), f(t), g(t), h(t)$ and $k(t)$ be real valued non-negative continuous functions defined on I; $G(u)$ be a continuous, strictly increasing, convex and submultiplicative function for $u \geqslant 0$; $G(0) = 0$, $\lim_{u \to \infty} G(u) = \infty$; $\alpha(t)$, $\beta(t)$ be continuous functions on I; $\alpha(t), \beta(t) > 0$, $\alpha(t) + \beta(t) = 1$; and

$$u(t) \leqslant f(t) + g(t)\, G^{-1}\Big[\int_0^t h(s)\, G(u(s))\, ds + \int_0^t h(s)\, \beta(s)\, G(g(s)\, \beta^{-1}(s)) \times \Big(\int_0^s k(\tau)\, G(u(\tau))\, d\tau\Big)\, ds\Big] \quad \ldots(1)$$

for all $t \in I$. Then

$$u(t) \leqslant f(t) + g(t)\, G^{-1}\Big[\int_0^t h(s)\, \{\alpha(s)\, G(f(s)\, \alpha^{-1}(s)) + \beta(s)\, G(g(s)\, \beta^{-1}(s)) \times \exp\Big(\int_0^s \beta(\tau)\, G(g(\tau)\, \beta^{-1}(\tau))\, (h(\tau) + k(\tau))\, d\tau\Big) \int_0^s \alpha(\tau)\, G(f(\tau)\, \alpha^{-1}(\tau))(h(\tau) + k(\tau)) \exp\Big(-\int_0^\tau \beta(\eta)\, G(g(\eta)\, \beta^{-1}(\eta))\, (h(\eta) + k(\eta))\, d\eta\Big)\, d\tau\}\, ds\Big] \quad \ldots(2)$$

for all $t \in I$.

PROOF : Rewrite (1) as

$$u(t) \leqslant \alpha(t)\, (f(t)\, \alpha^{-1}(t)) + \beta(t)\, (g(t)\, \beta^{-1}(t))\, G^{-1}\Big[\int_0^t h(s)\, G(u(s))\, ds + \int_0^t h(s)\, \beta(s)\, G(g(s)\, \beta^{-1}(s))\, \Big(\int_0^s k(\tau)\, G(u(\tau))\, d\tau\Big)\, ds\Big].$$

Since G is convex, submultiplicative and monotonic, we have

$$G(u(t)) \leqslant \alpha(t)\, G(f(t)\, \alpha^{-1}(t)) + \beta(t)\, G(g(t)\, \beta^{-1}(t))\, \Big[\int_0^t h(s)\, G(u(s))\, ds + \int_0^t h(s)\, \beta(s)\, G(g(s)\, \beta^{-1}(s))\, \Big(\int_0^s k(\tau)\, G(u(\tau))\, d\tau\Big)\, ds\Big].$$

Now an application of Theorem 1 yields the desired bound in (2). The proof of the theorem is complete.

By setting $G(u) = u^p$, $1 \leqslant p < \infty$, in Theorem 2, we arrive at the following interesting corollary.

Corollary — Let $u(t), f(t), g(t), h(t)$ and $k(t)$ be real valued non-negative continuous functions defined on I, let $\alpha(t)$, $\beta(t)$ be positive continuous functions on I such that $\alpha(t) + \beta(t) = 1$, let $1 \leqslant p < \infty$ and suppose that

$$u(t) \leqslant f(t) + g(t) \Big[\int_0^t h(s)\, u^p(s)\, ds + \int_0^t h(s)\, \beta(s)\, (g(s)\, \beta^{-1}(s))^p \times \Big(\int_0^s k(\tau)\, u^p(\tau)\, d\tau \Big)\, ds \Big]^{1/p}$$

for all $t \in I$. Then

$$u(t) \leqslant f(t) + g(t) \Big[\int_0^t h(s) \Big\{ \alpha(s)\, (f(s)\, \alpha^{-1}(s))^p + \beta(s)\, (g(s)\, \beta^{-1}(s))^p \times \exp\Big(\int_0^s \beta(\tau)\, (g(\tau)\, \beta^{-1}(\tau))^p\, (h(\tau) + k(\tau)) \Big) \int_0^s \alpha(\tau) (f(\tau)\, \alpha^{-1}(\tau))^p\, (h(\tau) + k(\tau))\, \exp\Big(- \int_0^\tau \beta(\eta)\, (g(\eta)\, \beta^{-1}(\eta))^p\, (h(\eta) + k(\eta))\, d\eta \Big)\, d\tau \Big\}\, ds \Big]^{1/p}$$

for all $t \in I$.

Finally, we state and prove the following generalization of the Gronwall-Bellman inequality which may be used in applications.

Theorem 3 — Let $u_i(t), f_i(t)$ be real valued non-negative continuous functions defined on I, and $a_i(t, s)$, $b_i(t, s)$ be continuous real valued functions defined on $I \times I \to R^+$, for which the inequality

$$u_i(t) \leqslant f_i(t) + \int_0^t a_i(t, s)\, u_i(s)\, ds + \int_0^t a_i(t, s) \Big(\int_0^s b_i(s, \tau)\, u_i(\tau)\, d\tau \Big)\, ds \qquad \ldots(3)$$

holds for all $t \in I$, where $i = 1, \ldots, n$. Define

$$g(t)\, h(s) = \max_{0 \leqslant s \leqslant t} a_i(t, s),\ i = 1, \ldots, n \qquad \ldots(4)$$

$$g(t)\, k(s) = \max_{0 \leqslant s \leqslant t} b_i(t, s),\ i = 1, \ldots, n \qquad \ldots(5)$$

$$f(t) = \sum_{i=1}^n f_i(t), \text{ for } t \in I. \qquad \ldots(6)$$

Then

$$\sum_{i=1}^{n} u_i(t) \leqslant f(t) + g(t) \left[\int_0^t h(s) \{ f(s) + g(s) \exp \left(\int_0^s g(\tau)(h(\tau) + k(\tau)) \right. \right.$$

$$\int_0^s f(\tau)(h(\tau) + k(\tau)) \exp \left(- \int_0^\tau g(\eta)(h(\eta) + k(\eta))\, d\eta \right) d\tau \Big) ds \Big] \qquad \ldots(7)$$

for all $t \in I$.

PROOF: Substituting $i = 1, \ldots, n$ in (3) and adding these inequalities we have

$$\sum_{i=1}^{n} u_i(t) \leqslant f(t) + g(t) \left[\int_0^t h(s) \sum_{i=1}^{n} u_i(s)\, ds + \int_0^t h(s)\, g(s) \left(\int_0^s k(\tau) \sum_{i=1}^{n} u_i(\tau)\, d\tau \right) ds \right]$$

in view of (4), (5) and (6). Now an application of Theorem 1 yields the desired bound in (7).

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