NUMERICAL SOLUTION OF NON-SINGULAR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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(Communicated by B. R. Seth, F.N.A.)

(*Received* 14 *March* 1973; *after revision* 9 *October* 1973)

In this paper, spline functions are employed for the numerical solution of non-singular Fredholm integral equations of the second kind. For comparison, two more methods, the trapezoidal and the Chebyshev series methods, are also described in brief. The three methods are applied to Love's equation and the numerical results show that the spline method is potentially useful.

1. INTRODUCTION

Consider the equation

$$y(x) + \int_{-1}^{1} K(x, s)\, y(s)\, ds = f(x) \qquad \ldots(1)$$

where $-1 \leqslant x,\ s \leqslant 1$. In this equation, which is known as Fredholm integral equation of the second kind, $f(x)$ is a given function and $y(x)$ is unknown. We consider here non-singular integral equations only, i.e., equations in which the kernel $K(x, s)$ is continuous and bounded. In §§2, 3 and 4, we describe three methods for the numerical solution of (1) and §5 gives a numerical example and results. The first two methods, viz., the trapezoidal and the Chebyshev series methods are well-known and are described here only briefly. The third one, employing spline functions, is a new method, and is, therefore, described here in more detail.

2. THE TRAPEZOIDAL METHOD

To solve (1) in the interval $(-1, 1)$, we divide it into smaller intervals of width h, the ith point of subdivision being denoted by s_i, such that

$$s_i = -1 + ih,\ i = 0, 1, 2, \ldots, N$$

and $Nh = 2$.

The approximate solution at these mesh points will be denoted by $\bar{y}_i$. Then, (1) may be written as :

$$\tilde{y}_i + \sum_{j=0}^{N-1} \int_{s_j}^{s_{j+1}} K(x_i, s)\, y(s)\, ds = f_i.$$

Approximating the integral term by the trapezoidal rule, this gives :

$$\tilde{y}_i + \sum_{j=0}^{N-1} \frac{h}{2} [K(x_i, s_j)\, \tilde{y}(s_j) + K(x_i, s_{j+1})\, \tilde{y}(s_{j+1})] = f_i$$

which can be written as

$$\tilde{y}_i + \frac{h}{2} K(x_i, s_0)\, \tilde{y}_0 + \frac{h}{2} K(x_i, s_n)\, \tilde{y}_n + h \sum_{j=1}^{N-1} K(x_i, s_j)\, \tilde{y}_j = f_i,$$

$$i = 0, 1, 2, \ldots, N \qquad \ldots(2)$$

It was shown by Linz (1969) that the trapezoidal quadrature scheme is convergent with order at least two.

3. Chebyshev Series Method

$$\text{Let } y(x) = \sum_{r=0}^{N} a_r T_r(x) \qquad \ldots(3)$$

and

$$f(x) = \sum_{r=0}^{N} f_r T_r(x) \qquad \ldots(4)$$

where $T_r(x)$ is the rth Chebyshev polynomial. Substituting (3) and (4) in (1), and interchanging the order of integration and summation in the integral term, we obtain

$$\sum_{r=0}^{N} a_r T_r(x) + \sum_{j=0}^{N} a_j \int_{-1}^{1} K(x, s)\, T_j(s)\, ds = \sum_{r=0}^{N} f_r T_r(x). \qquad \ldots(5)$$

If we can now find the expansion

$$\int_{-1}^{1} K(x, s)\, T_j(s)\, ds = \sum_{r=0}^{N} b_{jr} T_r(x) \qquad \ldots(6)$$

we can equate corresponding coefficients of each $T_r(x)$ on both sides of (5), and obtain a set of $(N + 1)$ equations in $(N + 1)$ unknowns a_r :

$$a_r + \sum_{j=0}^{N} a_j b_{jr} = f_r,\ r = 0, 1, 2, \ldots, N \qquad \ldots(7)$$

This method, which provides the solution as a Chebyshev expansion, is due to Elliott (1963). We describe here a variation of this method based on the Clenshaw and Curtis (1960) quadrature scheme which gives better accuracy and is due to El-Gendi (1969).

Consider the numerical quadrature of the definite integral

$$\int_{-1}^{1} f(x)\,dx \qquad \ldots(8)$$

where $f(x)$ is defined and well-behaved in the interval $-1 \leqslant x \leqslant 1$. The Clenshaw and Curtis quadrature is based on the approximation

$$f(x) = \sum_{r=0}^{N}{}'' a_r T_r(x) \qquad \ldots(9)$$

where

$$a_r = \frac{2}{N} \sum_{j=0}^{N}{}'' f\left(\cos \frac{j\pi}{N}\right) \cos \frac{rj\pi}{N} \qquad \ldots(10)$$

In the above, $T_r(x)$ is the rth Chebyshev polynomial, and the double primes denote that the first and last terms have to be halved.

Substituting (9) and (10) in (8), and simplifying using the relation given by Jain (1971) :

$$\int_{-1}^{1} T_{2j}(x)\,dx = \frac{2}{1-4j^2} \qquad \ldots(11)$$

We can write

$$\int_{-1}^{1} f(x)\,dx = \sum_{s=0}^{N} b_{Ns} f_s \qquad \ldots(12)$$

where for even N,

$$b_{NS} = \frac{4}{N} \sum_{j=0}^{N/2}{}'' \frac{1}{1-4j^2} \cos \frac{2j\pi s}{N}, \quad s = 1, 2, 3, \ldots, N-1. \qquad \ldots(13)$$

and

$$b_{NO} = b_{NN} = \frac{1}{N^2 - 1}. \qquad \ldots(14)$$

The integral term in (1) can also be approximated in the same way. Following the same procedure as above, (1) can be written in the matrix form :

$$[I + A]\,[\mathbf{y}] = [\mathbf{f}] \qquad \ldots(15)$$

where the elements of A are defined as

$$a_{ij} = b_{Nj} K_{ij} \quad ...(16)$$

$$K_{ij} = K\left(-\cos\frac{i\pi}{N}, -\cos\frac{j\pi}{N}\right), \quad ...(17)$$

$$i, j = 0, 1, 2, ..., N.$$

and

$$y_i = y\left(-\cos\frac{i\pi}{N}\right) \quad ...(18)$$

b_{Nj} being given by (13) and (14) above. The system (15) can now be solved to obtain directly the values of y.

Similar approximations of Volterra equations and integro-differential equations are given in the cited paper by El-gendi.

4. Method Using Cubic Spline Solutions

Several authors, e.g., Bickley (1968), Albasiny and Hoskins (1969), and Fyfe (1969), employed cubic splines to the solution of two-point boundary value problems. It does not appear that cubic splines are employed anywhere for the numerical solution of integral equations*. In this section, we describe a method using cubic spline approximation for the numerical solution of non-singular Fredholm integral equations of the second kind of type (1). The method produces the solution at certain points, called knots, and is more accurate than the methods previously described.

Interpolating Cubic Spline

Consider the problem of interpolating between given data points (x_i, y_i), $i = 0, 1, 2, ..., n$ with $x_0 < x_1 < x_2 ... < x_n$ by means of a function $S(x)$ which is such that $S \in C^2[x_0, x_n]$ and $\int_{x_0}^{x_n} [S''(x)]^2\, dx$ is a minimum. Ahlberg *et al.* (1967) have shown that there exists a unique S with these properties and that, in the interval $x_{j-1} \leqslant x \leqslant x_j$, $S(x)$ is given by

$$\begin{aligned} S(x) = {} & M_{j-1}\frac{(x_j - x)^3}{6h} + M_j\frac{(x - x_{j-1})^3}{6h} \\ & + \left(y_{j-1} - \frac{h^2}{6} M_{j-1}\right)\frac{(x_j - x)}{h} \\ & + \left(y_j - \frac{h^2}{6} M_j\right)\frac{(x - x_{j-1})}{h} \end{aligned} \quad ...(19)$$

*Phillips (1972) uses cubic spline collocation as a projection method for the solution of integral equations of the second kind.

where $M_j = S''(x_j)$, $y_j = y(x_j)$ and

$x_j = x_0 + jh \quad (j = 0, 1, 2, \ldots, n)$.

The first derivatives are given by

$$\left.\begin{aligned} S'(x_j+) &= -\frac{h}{3} M_j - \frac{h}{6} M_{j+1} + \frac{y_{j+1} - y_j}{h}, \\ &\qquad j = 0, 1, \ldots, n-1 \\ \text{and} \quad S'(x_j-) &= \frac{h}{3} M_j + \frac{h}{6} M_{j-1} + \frac{y_j - y_{j-1}}{h}, \\ &\qquad j = 1, 2, \ldots, n. \end{aligned}\right\} \quad \ldots(20)$$

so that their continuity implies

$$\frac{h}{6} M_{j-1} + \frac{2h}{3} M_j + \frac{h}{6} M_{j+1} = \frac{y_{j+1} - 2y_j + y_{j-1}}{h}, \quad j = 1, 2, 3, \ldots, n-1 \quad \ldots(21)$$

where

$$M_0 = M_n = 0. \quad \ldots(22)$$

Solution of the Integral Equation

The integral term in (1) can now be approximated by using (19). We have

$$\begin{aligned} y(x_i) + \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} K(x_i, s) \Bigg[& M_{j-1} \frac{(s_j - s)^3}{6h} + M_j \frac{(s - s_{j-1})^3}{6h} \\ & + \left(y_{j-1} - \frac{h^2}{6} M_{j-1} \right) \frac{(s_j - s)}{h} \\ & + \left(y_j - \frac{h^2}{6} M_j \right) \frac{(s - s_{j-1})}{h} \Bigg] ds \\ & = f(x_i), \; i = 0, 1, 2, \ldots, n. \end{aligned}$$

Putting $s = s_{j-1} + ph$, the above equation simplifies to

$$\begin{aligned} y(x_i) + h \sum_{j=1}^{n} \int_{0}^{1} K(x_i, s_{j-1} + ph) \Bigg[& M_{j-1} \frac{(1-p)^3 h^2}{6} + M_j \frac{p^3 h^2}{6} \\ & + \left(y_{j-1} - \frac{h^2}{6} M_{j-1} \right) (1 - p) \\ & + \left(y_j - \frac{h^2}{6} M_j \right) p \Bigg] dp \\ = f(x_i), & \\ i = 0, 1, 2, \ldots, n. & \end{aligned} \quad \ldots(23)$$

In (23), the integrals

$$\int_0^1 K(x_i, s_{j-1} + ph)\, p^m dp, \quad m = 0, 1, 2 \text{ and } 3 \qquad \ldots(24)$$

have to be evaluated. In the particular example chosen here, it was possible to evaluate them analytically. This is, of course, not always possible and numerical techniques may have to be employed. When these integrals are evaluated, (23) yields a system of $(n + 1)$ linear algebraic equations which together with (21) and (22) will form a set of $(2n + 2)$ unknowns, viz., $y_0, y_1, y_2, \ldots, y_n, M_0, M_1, M_2, \ldots, M_n$.

5. Numerical Example

The integral equation

$$y(x) + \int_{-1}^{1} K(x, s)\, y(s)\, ds = 1 \qquad \ldots(25)$$

where

$$K(x, s) = \frac{1}{\pi}\left[\frac{d}{d^2 + (x - s)^2}\right] \qquad \ldots(26)$$

and d is a positive real number, occurs in the problem of determining the capacity of a circular plate condenser and was considered by Love (1949). He showed, by analytical methods, that there exists a unique, continuous, real and even solution, and that it can be expressed as a convergent series of the form

$$y(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \int_{-1}^{1} K_n(x, s)\, ds \qquad \ldots(27)$$

where the iterated Kernels $K_n(x, s)$ are given by

$$\left.\begin{aligned} K_1(x, s) &= \frac{d}{\pi\,[d^2 + (x - s)^2]} \\ K_n(x, s) &= \int_{-1}^{1} K_{n-1}(x, t)\, K_1(t, s)\, dt. \end{aligned}\right\} \qquad \ldots(28)$$

This method of solution is somewhat laborious, and numerical solutions to this problem were found by several authors, e.g., Fox and Goodwin (1953), Young (1954), Elliott (1963), Wolfe (1969), El-Gendi (1969) and Phillips (1972). All these authors, excepting Phillips, investigated the problem only for the case $d = 1.0$. Phillips discusses the problem even for smaller values of d, which is more interesting. It should be remarked here that the trapezoidal and the Chebyshev series methods are easy of application only for the case $d = 1.0$, whereas the spline function method can be applied when the values of d are small. Table I gives the results for the trapezoidal method with $d = 1.0$.

Although the computations were made with $h = 0.0625$, only those values at $x = 0$ and $x = 1$ are given here in order to enable a direct comparison and estimation of error. The h^2-order of convergence is quite revealing.

TABLE I

Trapezoidal solution of Love's equation for $d = 1.0$

x	Accurate $y(x)$	N	Computed $y(x)$	Error	Ratio
0.0	0.65741	4	0.66026	0.00285	
		8	0.65812	0.00071	4
		16	0.65759	0.00018	4
		32	0.65746	0.00005	3.6
1.0	0.75572	4	0.75452	0.00120	
		8	0.75542	0.00030	4
		16	0.75564	0.00008	3.75
		32	0.75570	0.00002	4

As was already remarked above, this method is unsuitable for smaller values of d. Thus, for example with 32 subdivisions and $d = 0.001$, the value obtained for $x = 0$ is 0.04782 compared to the true value 0.50015.

Table II summarizes the results obtained by the Chebyshev series method with $N = 8$ and $d = 1.0$, and it is clear that this method gives better accuracy than the trapezoidal method. For the sake of comparison, the Chebyshev coefficients were also computed. These agree quite well with those obtained by El-Gendi, and are given in Table III. It should be sufficient to remark here that this method too gives inaccurate results for smaller values of d.

TABLE II

Chebyshev series solution of Love's equation with $d = 1.0$

$x_j = -\cos \frac{j\pi}{N}$	$y(x_j)$
0.0	0.65740981
0.38268	0.67248912
0.70711	0.70866017
0.92388	0.74265684
1.0	0.75571801

TABLE III

The Chebyshev coefficients a_r

r	a_{2r}
0	1.4151850
1	0.0493851
2	−0.0010481
3	−0.0002310
4	0.0000391

For the spline function method, the integrals in (24) were calculated analytically. Thus for $m = 0$, we have

$$X_0 = \int_0^1 K(x_i, s_{j-1} + ph)\, dp$$

$$= \frac{1}{\pi} \int_0^1 \frac{d}{d^2 + (x_i - s_{j-1} - ph)^2}\, dp$$

Putting $x_i = -1 + ih$ and $s_{j-1} = -1 + \overline{j-1}\, h$, and evaluating the definite integral, we obtain

$$X_0 = \frac{1}{h\pi} \tan^{-1} \frac{h/d}{1 + \frac{h^2}{d^2}(i-j)(i-j+1)}. \qquad \text{...(29)}$$

Similarly, we obtain the results

$$X_1 = \int_0^1 K(x_i, s_{j-1} + ph)\, p\, dp$$

$$= \frac{d}{2\pi h^2} \left[\log \frac{d^2 + h^2(i-j)^2}{d^2 + h^2(i-j+1)^2}\right] + (i-j+1)\, X_0 \qquad \text{...(30)}$$

$$X_2 = \int_0^1 K(x_i, s_{j-1} + ph)\, p^2\, dp$$

$$= \frac{d}{\pi h^2} - \left(\frac{d^2}{h^2} + \overline{i-j+1}^2\right) X_0 + 2X_1(i-j+1) \qquad \text{...(31)}$$

and $$X_3 = \int_0^1 K(x_i, s_{j-1} + ph)\, p^3\, dp$$

$$= \frac{d}{2\pi h^2}\,[5 + 4(i - j)] + \left[3(i - j + 1)^2 - \frac{d^2}{h^2}\right] X_1$$

$$- 2(i - j + 1)\left\{\frac{d^2}{h^2} + (i - j + 1)^2\right\} X_0 \qquad \ldots(32)$$

The system of equations was solved by the Gauss-Seidel iteration method and a standard subroutine was used for this. The results are summarized in Table IV for different values of d, and agree closely well with those obtained by Phillips.

It was found that the method is unsuitable for finding the solution for larger values of d as the convergence is rather slow. Thus, for $d = 1.0$ the value obtained with 500 iterations for $x = 1.0$ is 0.80692 compared to the true value 0.75572.

TABLE IV

Cubic spline solutions of Love's equation

x	$y(x)$		
	$d = 0.1$	$d = 0.01$	$d = 0.001$
0.0	0.51261	0.50146	0.50015
0.1	0.51353	0.50150	0.50015
0.2	0.51470	0.50158	0.50016
0.3	0.51629	0.50170	0.50017
0.4	0.51858	0.50187	0.50019
0.5	0.52216	0.50214	0.50022
0.6	0.52876	0.50261	0.50026
0.7	0.54630	0.50322	0.50035
0.8	0.60688	0.51713	0.50271
0.9	0.68960	0.53902	0.50713
1.0	0.78627	0.69641	0.67179

It is known that the capacity of the condenser is given by

$$C = \frac{1}{\pi}\int_0^1 y(t)\, dt.$$

The values of the integral $\int_0^1 y(t)\, dt$ are calculated by Simpson's rule and are given in Table V.

TABLE V

Values of the integral $I = \int_0^1 y(t)\,dt$

d	I
0.001	0.50702
0.01	0.51449
0.1	0.55961
1.0	0.63303

It is interesting to note that the value obtained by Cooke as quoted by Sneddon (1966) for $d = 1.0$ is 0.6912 and that the values of the integral for smaller values of d are approaching its true value for $d = 0.0$ viz., 0.5.

6. Conclusion

The numerical results show that the spline method is potentially useful. Its application to more complicated problems will have to be examined together with an estimation of error in the method. It seems probable that the condition of continuity of the kernel may be relaxed, and the advantage to be achieved by using unequal intervals may also be explored. The solution obtained by the spline method can be improved upon by regarding it as the initial iterate in an iterative method of higher order of convergence. These suggestions have not been implemented since it is the primary purpose here to show that spline functions can be employed successfully for the numerical solution of Fredholm integral equations of the second kind.

Acknowledgement

The author thanks the referee for his most helpful comments which have contributed towards an improvement of the paper.

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